THE GROUND STATE ENERGY OF THE THREE DIMENSIONAL GINZBURG-LANDAU FUNCTIONAL PART II: SURFACE REGIME

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Abstract. We study the Ginzburg-Landau model of superconductivity in three dimensions and for strong external magnetic fields. For magnetic field strengths above the phenomenologically defined second critical field it is known from Physics that superconductivity should be essentially restricted to a region near the boundary. We prove that the expected region does indeed carry superconductivity. Furthermore, we give precise energy estimates valid also in the regime around the second critical field which display the transition from bulk superconductivity to surface superconductivity.

1. Introduction and main results

The phenomenological Ginzburg-Landau theory of superconductivity successfully describes the behavior of a superconductor subject to an external magnetic field. Also, as pointed out in the celebrated work of Abrikosov, this theory predicted the existence of type II superconductors before they had been empirically realized, see [15] for a review of this physical topic for which A. Abrikosov was awarded the Nobel Prize.

In the Ginzburg-Landau theory, the superconducting state of a sample is described by a complex-valued wave function ψ and a vector field (magnetic potential) $\mathbf{A}$ such that the pair $(\psi, \mathbf{A})$ is a critical point of a specific energy (see (1.1) below). The physical interpretation of ψ and $\mathbf{A}$ is explained by the microscopic Bardeen-Cooper-Schrieffer (BCS) theory as follows: $|\psi|^2$ is proportional to the density of superconducting particles and $\operatorname{curl} \mathbf{A}$ measures the induced magnetic filed inside the sample. The rigorous mathematical justification of the connection between the Ginzburg-Landau and the BCS theory has only been established recently in [14].

The behavior of a type II superconductor is distinguished by three critical values the intensity of the applied magnetic field can have, that we denote by H_{C_1}, H_{C_2} and H_{C_3}. These critical fields may be described in terms of the wave function ψ as follows. Suppose H is the intensity of the external magnetic field applied to the sample. If $H < H_{C_1}$, the material is in the superconducting phase, which corresponds to $|\psi| > 0$ everywhere. If $H_{C_1} < H < H_{C_2}$, the magnetic field penetrates the sample in quantized vortices (corresponding to zeros of ψ). If $H_{C_2} < H < H_{C_3}$, superconductivity is confined to (part of) the surface of the sample (corresponding to $|\psi|$ very small in the bulk). Finally, if $H > H_{C_3}$, superconductivity is lost, which is reflected by $\psi = 0$ everywhere.

In the last two decades, much progress has been made in order to establish the aforementioned behavior of Type II superconductors by studying minimizers of the Ginzburg-Landau energy. The monograph [24] and references therein contains an analysis of vortices and the critical field H_{C_1}. Concerning the analysis of the critical fields H_{C_2} and H_{C_3} we mention [13, 8] (and references therein). As one can see in [8, 24], the Ginzburg-Landau model has a rich mathematical structure whose analysis requires a diversity of methods, and many of them have been developed especially for the study of the model.

While a detailed study of the Ginzburg-Landau model in a two dimensional domain has been the subject of numerous papers, the study of the model in a three dimensional domain is much

Key words and phrases. Ginzburg-Landau functional, thermodynamic limits, elliptic estimates, variational methods, magnetic Schrödinger operators, semiclassical analysis.

less developed. Among the important problems that are still open in 3D is the calculation of the critical field H_{C_1} for general domains [1]. Also, when comparing with the existing results for 2D domains, a precise localization of the wave-function ψ is absent when the external applied magnetic field varies from H_{C_1} up to H_{C_3}. However, for both 2D and 3D domains, a sharp characterization of the critical field H_{C_3} is given in [10]. In this paper, together with [12, 19], we give a detailed description of the behavior of the wave-function ψ and its energy for external magnetic fields varying in the range above H_{C_1} and up to H_{C_3}. The results in this paper concern surface (3D) superconductivity and the transition that happens close to H_{C_2} from bulk to surface superconductivity.

Let $\Omega \subset \mathbb{R}^3$ be a bounded and open set with smooth boundary which models a superconducting sample subject to an applied external magnetic field. The energy of the sample is given by the Ginzburg-Landau functional,

$$\mathcal{E}^{3\mathrm{D}}(\psi,\mathbf{A}) = \mathcal{E}^{3\mathrm{D}}_{\kappa,H}(\psi,\mathbf{A}) = \int_\Omega \Big[|(\nabla - i\kappa H\mathbf{A})\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4\Big]\,dx + \kappa^2H^2\int_{\mathbb{R}^3}|\operatorname{curl}\mathbf{A}-\beta|^2\,dx\,. \tag{1.1}$$

Here κ and H are two positive parameters whose physical interpretation are as follows, the number κ is a material parameter, and the number H is the intensity of a constant magnetic field externally applied to the sample. As explained earlier, ψ is a wave function (order parameter) and $\mathbf{A}$ is the induced magnetic potential. We take ψ and $\mathbf{A}$ in convenient spaces as follows,

$$\psi \in H^1(\Omega;\mathbb{C})\,,\quad \mathbf{A}\in \dot H^1_{\mathrm{div},\mathbf{F}}(\mathbb{R}^3)\,,$$

where $\dot H^1_{\mathrm{div},\mathbf{F}}(\mathbb{R}^3)$ is the space introduced in (1.2) below. Finally, β is the profile of the external magnetic field that we choose constant, $\beta = (0,0,1)$.

Let $\dot H^1(\mathbb{R}^3)$ be the homogeneous Sobolev space, i.e. the closure of $C_c^\infty(\mathbb{R}^3)$ under the norm $u \mapsto \|u\|_{\dot H^1(\mathbb{R}^3)} := \|\nabla u\|_{L^2(\mathbb{R}^3)}$. Let further $\mathbf{F}(x) = (-x_2/2, x_1/2, 0)$. Clearly $\operatorname{div}\mathbf{F} = 0$.

We define the space,

$$\dot H^1_{\mathrm{div},\mathbf{F}}(\mathbb{R}^3) = \{\mathbf{A}\ :\ \operatorname{div}\mathbf{A} = 0\,,\quad \text{and}\quad \mathbf{A}-\mathbf{F}\in \dot H^1(\mathbb{R}^3)\}\,. \tag{1.2}$$

Critical points $(\psi,\mathbf{A}) \in H^1(\Omega;\mathbb{C})\times \dot H^1_{\mathrm{div},\mathbf{F}}(\mathbb{R}^3)$ of $\mathcal{E}^{3\mathrm{D}}$ satisfy the Ginzburg-Landau equations,

$$\begin{cases} -(\nabla - i\kappa H\mathbf{A})^2\psi = \kappa^2(1-|\psi|^2)\psi & \text{in}\quad \Omega,\\ \operatorname{curl}^2\mathbf{A} = -\dfrac{1}{\kappa H}\Im(\overline{\psi}\,(\nabla - i\kappa H\mathbf{A})\psi)\mathbf{1}_\Omega & \text{in}\quad \mathbb{R}^3,\\ N\cdot(\nabla - i\kappa H\mathbf{A})\psi = 0 & \text{on}\quad \partial\Omega\,, \end{cases} \tag{1.3}$$

where $\mathbf{1}_\Omega$ is the characteristic function of the domain Ω, and N is the interior unit normal vector of $\partial\Omega$.

For a solution $(\psi,\mathbf{A})$ of (1.3), the function ψ describes the superconducting properties of the material and $H\operatorname{curl}\mathbf{A}$ gives the induced magnetic field.

The important class materials called Type II superconductors corresponds mathematically to the limit $\kappa\to\infty$, see [8, 24].

We define the ground state energy,

$$\mathrm{E}_{\mathrm{g.st}}(\kappa,H) = \inf_{(\psi,\mathbf{A})\in H^1(\Omega;\mathbb{C})\times \dot H^1_{\mathrm{div},\mathbf{F}}(\mathbb{R}^3)} \mathcal{E}^{3\mathrm{D}}(\psi,\mathbf{A})\,. \tag{1.4}$$

The leading order asymptotics of the ground state energy involves a function g constructed in [12] (the definition will be recalled in Section 2) satisfying that $g : [0,\infty) \to [-1/2, 0]$ is

[1] In a recent paper [18], compactness results for the 3D functional valid in general domain are obtained, which allow to obtain the leading order term of H_{C_1}. Earlier results include a candidate for the expression of H_{C_1} in the case of the ball [2], and an expression of H_{C_1} in 'thin' shell domains in [6].

continuous, increasing and there is a constant $E_2 < 0$ with

$$g(b) = 0, \qquad \forall\, b \geq 1, \tag{1.5}$$

$$g(b) = E_2(1-b)^2[1+o(1)], \qquad \text{as } b \nearrow 1. \tag{1.6}$$

From [12] we have the following general (bulk) result as long as H/κ is bounded from below

$$\left| \mathrm{E}_{\mathrm{g.st}}(\kappa, H) - g(\tfrac{H}{\kappa})|\Omega|\kappa^2 \right| \leq C\kappa^{3/2}. \tag{1.7}$$

However, when $H/\kappa \geq 1$, $g(H/\kappa) = 0$ and (1.7) only gives a somewhat weak estimate of the energy. The value $H = \kappa$ corresponds to the phenomenologically described critical field H_{C_2} mentioned previously. So one expects that around this value superconductivity should become concentrated near the boundary of the domain Ω and this should be reflected in the energy asymptotics.

In this paper we will complete the study of the energy asymptotics for high magnetic field initiated in [12] by

- giving the leading order energy asymptotics for $H > \kappa$, which will be a surface energy of order of magnitude κ,
- studying the transition from 'bulk' to 'surface' dominated energy at $H \approx \kappa$. We will see that this transition takes place at $H - \kappa$ of order $\sqrt{\kappa}$ in the sense that
  - If $H < \kappa - f(\kappa)$, where $f(\kappa)/\sqrt{\kappa} \to +\infty$ as $\kappa \to \infty$, then the leading contribution to the energy comes from the bulk.
  - If $H > \kappa - f(\kappa)$, where $\limsup_{\kappa\to\infty} f(\kappa)/\sqrt{\kappa} \leq 0$ then the leading contribution to the energy comes from the surface.
  - If $H = \kappa - a\sqrt{\kappa}$ for some constant $a > 0$, then the bulk and surface contributions of the energy have the same order of magnitude.

We now state the main results of the paper, Theorems 1.1 and 1.2. These require the introduction of some notation.

- If x is a point on the boundary of Ω, then $\nu(x)$ denotes the angle in $[0, \pi/2]$ between the vector $\beta = (0,0,1)$ and the tangent plane to $\partial\Omega$ at the point x.
- If $\nu \in [0, \pi/2]$, $\zeta(\nu)$ is the lowest eigenvalue of a magnetic Schrödinger operator in the half-space, see Section 3.2. The function ζ is a continuous and strictly increasing bijection from $[0, \pi/2]$ to $[\Theta_0, 1]$ where $\Theta_0 \approx 0.59$ is a universal constant (the definition of Θ_0 will be recalled in (3.1) below).
- If $\nu \in [0, \pi/2]$ and $\mathfrak{b} \leq 1$, the constant $E(\mathfrak{b}, \nu)$ will be introduced in Section 3.3.3. $E(\mathfrak{b}, \nu)$ depends continuously on both $\mathfrak{b}$ and ν, vanishes when $\zeta(\nu) \geq \mathfrak{b}$, and $E(\mathfrak{b}, \nu) < 0$ otherwise.

Below is a statement of the main result concerning the asymptotic behavior of the ground state energy.

Theorem 1.1. *Let $\mu : \mathbb{R}_+ \to \mathbb{R}$ be a function satisfying $\lim\limits_{\kappa\to\infty} \mu(\kappa) = 0$. Then there exists a positive constant κ_0, and a function* $\mathrm{err} : \mathbb{R}_+ \to \mathbb{R}$ *such that $\lim\limits_{\kappa\to\infty} \mathrm{err}(\kappa) = 0$ and the following is true. If $\kappa \geq \kappa_0$ and $H \geq \kappa - \mu(\kappa)\kappa$, then the ground state energy in* (1.4) *satisfies,*

$$\mathrm{E}_{g.st}(\kappa, H) = \sqrt{\kappa H} \int_{\partial\Omega} E\left(\mathfrak{b}, \nu(x)\right) d\sigma(x) + E_2|\Omega|\, [\kappa - H]_+^2 + \mathrm{err}(\kappa) \max\left(\kappa, [\kappa - H]_+^2\right) .$$

Here $\mathfrak{b} = \min\left(\kappa/H, 1\right)$, and $d\sigma(x)$ is the surface measure on the boundary of Ω.

The next theorem concerns the behavior of order parameters. We use the convention that a subset $D \subset \Omega$ is smooth if there exists an open subset $\widetilde{D} \subset \mathbb{R}^3$ having a smooth boundary and such that $D = \widetilde{D} \cap \Omega$.

Theorem 1.2. *Let $\mu:\mathbb{R}_+\to\mathbb{R}$ be a function satisfying $\lim_{\kappa\to\infty}\mu(\kappa)=0$ and let $D\subset\Omega$ be smooth. Then there exist a positive constant κ_0, and a function* $\mathrm{err}:\mathbb{R}_+\to\mathbb{R}$ *such that* $\lim_{\kappa\to\infty}\mathrm{err}(\kappa)=0$, *and if $\kappa\geq\kappa_0$ and $H\geq\kappa-\mu(\kappa)\kappa$, then the following is true.*

(1) *If $(\psi,\mathbf{A})\in H^1(\Omega;\mathbb{C})\times\dot H^1_{\mathrm{div},\mathbf{F}}(\mathbb{R}^3)$ is a solution of (1.3), then,*

$$\frac{1}{2}\int_D|\psi|^4\,dx\leq-\kappa^{-1}\sqrt{\frac{H}{\kappa}}\int_{\partial\Omega\cap\overline{D}}E\left(\frac{\kappa}{H},\nu(x)\right)d\sigma(x)-E_2|D|\left[\frac{\kappa}{H}-1\right]_+^2+\mathrm{err}(\kappa)\max\left(\frac{1}{\kappa},\left[\frac{\kappa}{H}-1\right]_+^2\right).\tag{1.8}$$

(2) *If $(\psi,\mathbf{A})\in H^1(\Omega;\mathbb{C})\times\dot H^1_{\mathrm{div},\mathbf{F}}(\mathbb{R}^3)$ is a minimizer of (1.1), then,*

$$\frac{1}{2}\int_D|\psi|^4\,dx=-\kappa^{-1}\sqrt{\frac{H}{\kappa}}\int_{\partial\Omega\cap\overline{D}}E\left(\frac{\kappa}{H},\nu(x)\right)d\sigma(x)-E_2|D|\left[\frac{\kappa}{H}-1\right]_+^2+\mathrm{err}(\kappa)\max\left(\frac{1}{\kappa},\left[\frac{\kappa}{H}-1\right]_+^2\right).\tag{1.9}$$

Remark 1.3. [Triviality for very large magnetic fields]
Many previous works have addressed the question of the third critical field, see [8] and the references therein for a detailed review of this topic. We define

$$H_{C_3}(\kappa)=\inf\big\{H>0,\text{ for all }H'>H\text{ the GL equations (1.3) have only trivial solutions}\big\}.\tag{1.10}$$

Here a solution is trivial if $\psi\equiv0$. It follows from [8] that

$$H_{C_3}(\kappa)=\Theta_0^{-1}\kappa+o(\kappa),\qquad\text{as }\kappa\to\infty.$$

Here, as we mentioned earlier, $\Theta_0\approx0.59$ is a universal constant whose definition will be recalled in (3.1) below. So we can conclude that the ratio H/κ when $\kappa\geq1$ is always uniformly bounded from above in the regime where non-trivial solutions of (1.3) exist.

It was realized early in the Physics literature (see for example [15, Chapter 6.6]) that when studying the 3-dimensional Ginzburg-Landau model between H_{C_2} and H_{C_3} there is a geometric boundary phenomenon that does not appear in 2D. When $\mathfrak{b}=\kappa/H\in(\Theta_0,1)$ (and κ sufficiently large) superconductivity will in the 2D case exist on the entire boundary. But in the 3D model only a certain part of the boundary will carry superconductivity. This part can be described using the angle $\nu(x)$ (between the magnetic field and the tangent plane at the boundary point x) and the spectral function ζ. A local and linearized calculation suggests that superconductivity should only be present near the boundary section

$$\Gamma(\mathfrak{b}):=\{x\in\partial\Omega\,:\,\zeta(\nu(x))<\mathfrak{b}\}.\tag{1.11}$$

In previous works, notably [20], the technique of Agmon estimates was used to prove that the superconducting order parameter ψ will indeed decay rapidly away from $\Gamma(\mathfrak{b})$. Our results give the opposite direction, namely exhibit through Theorem 1.2 that minimizers are indeed nonzero (in an L^4-sense) exactly in the vicinity of $\Gamma(\mathfrak{b})$. In order to realize this, notice that $[\kappa/H-1]_+=0$ by the condition on $\mathfrak{b}$ and that $E(\mathfrak{b},\nu)=0$ if and only if $\mathfrak{b}\leq\zeta(\nu)$.

2. The universal constant E_2

Given a constant $b\geq0$ and an open set $\mathcal{D}\subset\mathbb{R}^2$, we define the following Ginzburg-Landau energy,

$$G_{\mathcal{D}}(u)=\int_{\mathcal{D}}\Big(b|(\nabla-i\mathbf{A}_0)u|^2-|u|^2+\frac{1}{2}|u|^4\Big)\,dx\,.\tag{2.1}$$

Here $\mathbf{A}_0$ is the canonical magnetic potential,

$$\mathbf{A}_0(x_1,x_2)=\frac{1}{2}(-x_2,x_1)\,,\quad \forall\, x=(x_1,x_2)\in\mathbb{R}^2\,. \tag{2.2}$$

Given $R>0$, we denote by $K_R=(-R/2,R/2)\times(-R/2,R/2)$ a square of side length R. Let,

$$m_0(b,R)=\inf_{u\in H^1_0(K_R;\mathbb{C})} G_{K_R}(u)\,. \tag{2.3}$$

It is proved in [12] that

- For all $b>0$, there exists a constant $g(b)$ such that,
$$\lim_{R\to\infty}\frac{m_0(b,R)}{R^2}=g(b)\,.$$
- The function $\dfrac{g(b)}{(b-1)^2}$ has a limit as $b\to 1_-$,
$$E_2=\lim_{b\to 1_-}\frac{g(b)}{(b-1)^2}\,, \tag{2.4}$$
and $-\frac{1}{2}\leq E_2<0$.

3. Reduced Ginzburg-Landau energy

3.1. **Harmonic oscillator on the half-axis.** We denote by $\mathbb{R}_+=\{t\in\mathbb{R}\ :\ t>0\}$. For each real number $\xi\in\mathbb{R}$, we consider the harmonic oscillator

$$H(\xi)=-\frac{d^2}{dt^2}+(t-\xi)^2\quad\text{in}\quad L^2(\mathbb{R}_+)\,,$$

with Neumann boundary condition, $u'(0)=0$. Let $\mu_1(\xi)$ denotes the first eigenvalue of $H(\xi)$. We define the universal constant,

$$\Theta_0=\inf_{\xi\in\mathbb{R}}\mu_1(\xi)\,. \tag{3.1}$$

The constant Θ_0 satisfies the following properties [7, 17]:

$$\frac{1}{2}<\Theta_0<1,\quad\text{and}\quad \Theta_0=\mu_1(\xi_0),\quad\text{where}\quad \xi_0=\sqrt{\Theta_0}\,. \tag{3.2}$$

We introduce the function $\varphi_0\in L^2(\mathbb{R}^2)$ as follows,

$$\int_{\mathbb{R}_+}|\varphi_0(t)|^2\,dt=1\,,\quad -\varphi_0''(t)+(t-\xi_0)^2\varphi_0(t)=\Theta_0\varphi_0(t)\quad\text{in }\mathbb{R}_+\,,\quad \varphi_0'(0)=0\,. \tag{3.3}$$

3.2. **Magnetic Schrödinger operator in $\mathbb{R}^3_+$.** We denote by $\mathbb{R}^3_+=\{(x_1,x_2,x_3)\in\mathbb{R}^3\ :\ x_1>0\}$. Let $\nu\in[0,\pi/2]$. Consider the magnetic potential

$$\mathbf{E}_\nu(x)=(0,0,\cos\nu\,x_1+\sin\nu\,x_2)\,,\quad x=(x_1,x_2,x_3)\in\mathbb{R}^3_+\,. \tag{3.4}$$

Notice that $\operatorname{curl}\mathbf{E}_\nu=(\sin\nu,-\cos\nu,0)$ is constant. Geometrically, ν measures the angle between $\operatorname{curl}\mathbf{E}_\nu$ and the boundary of $\mathbb{R}^3_+$. Consider the Schrödinger operator with constant magnetic field,

$$\mathcal{L}(\nu)=-(\nabla-i\mathbf{E}_\nu)^2\quad\text{in}\quad L^2(\mathbb{R}^3_+)\,, \tag{3.5}$$

with domain

$$D(\mathcal{L}(\nu))=\{u\in L^2(\mathbb{R}^3_+)\ :\ (\nabla-i\mathbf{E}_\nu)^j u\in L^2(\mathbb{R}_+)\,,\ j=1,2\,,\ \partial_{x_1}u=0\text{ on }\{0\}\times\mathbb{R}^2\}\,.$$

We denote by $\zeta(\nu)$ the bottom of the spectrum of $\mathcal{L}(\nu)$:

$$\zeta(\nu)=\inf\sigma\big(\mathcal{L}(\nu)\big)\,. \tag{3.6}$$

We collect below some properties concerning $\zeta(\nu)$ (see e.g. [8, Lemmas 7.2.1 & 7.2.2]).

Lemma 3.1. *Let Θ_0 be the universal constant introduced in* (3.1) *above.*

(1) *The function $[0,\pi/2]\ni\nu\mapsto\zeta(\nu)$ is monotone increasing, $\zeta(0)=\Theta_0$ and $\zeta(\pi/2)=1$.*

(2) *For all* $\nu \in [0, \pi/2)$, $\zeta(\nu) < 1$.

We denote by $\mathbb{R}^2_+ = (0, \infty) \times \mathbb{R}$. In connection with the analysis of the operator $\mathcal{L}(\nu)$, we introduce the two dimensional operator

$$L(\nu) = -\partial_{x_1}^2 - \partial_{x_2}^2 + (\cos\nu\, x_1 + \sin\nu\, x_2)^2 \quad \text{in} \quad L^2(\mathbb{R}^2_+)\,, \tag{3.7}$$

whose domain is

$$D(L(\nu)) = \{u \in H^2(\mathbb{R}^2_+) \ :\ (\cos\nu\, x_1 + \sin\nu\, x_2)^j u \in L^2(\mathbb{R}^2_+),\ j = 1, 2,\ \partial_{x_1} u = 0 \ on\ \{0\} \times \mathbb{R}\}\,.$$

The link between the spectra of $\mathcal{L}(\nu)$ and $L(\nu)$ is given below [8].

Lemma 3.2. *Suppose* $\nu \in (0, \pi/2)$. *Then,*

(1) $\sigma(\mathcal{L}(\nu)) = \sigma(L(\nu))$ *;*
(2) $\sigma_{ess}(L(\nu)) = [1, \infty)$ *.*

Remark 3.3. Suppose that $\nu \in (0, \pi/2)$. It results from Lemma 3.2 and the properties of $\zeta(\nu)$ in Lemma 3.1 that

(1) $\zeta(\nu)$ is an eigenvalue of finite multiplicity of $L(\nu)$;
(2) $\zeta(\nu)$ is the lowest eigenvalue of $L(\nu)$.

Consequently, we can select a real-valued eigenfunction $\phi \in L^2(\mathbb{R}^2_+)$ such that:

$$\phi > 0\,, \quad \int_{\mathbb{R}^2_+} |\phi|^2\, dx = 1\,, \quad \int_{\mathbb{R}^2_+} \left(|\nabla\phi|^2 + |(\cos\nu\, x_1 + \sin\nu\, x_2)\phi|^2\right) dx = \zeta(\nu)\,.$$

Moreover, using the technique of 'Agmon estimates' (cf. [17]), it is proved in [5, Theorem 1.1] that the eigenfunction ϕ decays exponentially at infinity as follows. If $\nu \in (0, \pi/2)$ and $\alpha \in (0, \sqrt{1 - \zeta(\nu)})$, then there exists a positive constant $C_{\nu,\alpha}$ such that

$$\int_{\mathbb{R}^2_+} e^{\alpha\sqrt{x_1^2 + x_2^2}} \left(|\partial_{x_1}\phi|^2 + |\partial_{x_2}\phi|^2 + |(\cos\nu x_1 + \sin\nu x_2)\phi|^2 + |\phi|^2\right) dx_1 dx_2 \le C_{\nu,\alpha}. \tag{3.8}$$

3.3. **Reduced Ginzburg-Landau energy.** Let $\nu \in [0, \pi/2]$ and $\mathbf{E}_\nu$ be the magnetic potential introduced in (3.4). For each $\ell > 0$, we introduce the domains,

$$K_\ell = (-\ell, \ell) \times (-\ell, \ell)\,, \quad U_\ell = (0, \infty) \times K_\ell\,, \tag{3.9}$$

and the space,

$$\mathcal{S}_\ell = \{u \in L^2(U_\ell) \ :\ (\nabla - i\mathbf{E}_\nu)u \in L^2(U_\ell)\,,\ u = 0 \text{ on } (0, \infty) \times \partial K_\ell\}\,. \tag{3.10}$$

Let $\mathfrak{b} \in (0, 1]$ be a given constant. If $u \in \mathcal{S}_\ell$, we define the Ginzburg-Landau functional,

$$\mathcal{G}_{\mathfrak{b},\nu;\ell}(u) = \int_{U_\ell} \left(|(\nabla - i\mathbf{E}_\nu)u|^2 - \mathfrak{b}|u|^2 + \frac{\mathfrak{b}}{2}|u|^4\right) dx\,. \tag{3.11}$$

Associated with $\mathcal{G}_{\mathfrak{b},\nu;\ell}$ is the ground state energy,

$$d(\mathfrak{b}, \nu; \ell) = \inf_{u \in \mathcal{S}_\ell} \mathcal{G}_{\mathfrak{b},\nu;\ell}(u)\,. \tag{3.12}$$

3.3.1. *Preliminary properties of minimizers.*

Lemma 3.4. *For each* $\nu \in [0, \pi/2]$, *let* $\zeta(\nu)$ *be as defined in* (3.6). *Let* $\ell > 0$ *be given.*
If $\mathfrak{b} \le \zeta(\nu)$, *then*

$$d(\mathfrak{b}, \nu; \ell) = 0\,,$$

where $d(\mathfrak{b}, \nu; \ell)$ *is the ground state energy introduced in* (3.12).

Proof. By using $u = 0$ as a test function, it is clear that $d(\mathfrak{b}, \nu; \ell) \le 0$. On the other hand, the min-max principle and the condition on b show that $d(\mathfrak{b}, \nu; \ell) \ge 0$. □

Remark 3.5. It results from Lemma 3.4 and the properties of $\zeta(\nu)$ in Lemma 3.1 that $d(\mathfrak{b}, \nu; \ell) = 0$ in each of the following cases:

(1) $\mathfrak{b} \le \Theta_0$ and $\nu \in [0, \pi/2]$;
(2) $\mathfrak{b} \in [\Theta_0, 1]$ and $\nu = \pi/2$.

Theorem 3.6. *Let Θ_0 be the constant introduced in (3.1). Suppose that $\mathfrak{b} \in [\Theta_0, 1]$ is a given constant.*

For all $\nu \in [0, \pi/2]$ and $\ell \ge 1$, the functional $\mathcal{G}_{\mathfrak{b},\nu;\ell}$ in (3.11) admits a minimizer $\varphi_{\mathfrak{b},\nu;\ell} \in \mathcal{S}_\ell$ satisfying,

$$\mathcal{G}_{\mathfrak{b},\nu;\ell}(\varphi_{\mathfrak{b},\nu;\ell}) = d(\mathfrak{b}, \nu; \ell), \quad \|\varphi_{\mathfrak{b},\nu;\ell}\|_{L^\infty(U_\ell)} \le 1. \tag{3.13}$$

Furthermore, there exists a universal constant $C > 0$ such that, if $\nu \in [0, \pi/2]$ and $\ell > 0$, the minimizer $\varphi_{\mathfrak{b},\nu;\ell}$ satisfies,

$$\int_{U_\ell \cap \{x_1 \ge 4\}} \frac{x_1}{(\ln x_1)^2} \left(|(\nabla - i\mathbf{E}_\nu)\varphi_{\mathfrak{b},\nu;\ell}|^2 + |\varphi_{\mathfrak{b},\nu;\ell}|^2 + x_1^2 |\varphi_{\mathfrak{b},\nu;\ell}|^4 \right) dx \le C\ell^2. \tag{3.14}$$

Proof. Let $m > 0$ and consider the energy functional,

$$\mathcal{G}_{\ell,m}(u) = \int_{U_{\ell,m}} \left(|(\nabla - i\mathbf{E}_\nu)u|^2 - \mathfrak{b}|u|^2 + \frac{\mathfrak{b}}{2}|u|^4 \right) dx, \tag{3.15}$$

where $\mathbf{E}_\nu$ is the magnetic potential in (3.4) and,

$$U_{\ell,m} = (0, m) \times K_\ell, \quad K_\ell = (-\ell, \ell) \times (-\ell, \ell).$$

We define the ground state energy,

$$t(\ell, m) = \inf_{u \in \mathcal{S}_{\ell,m}} \mathcal{G}_{\ell,m}(u), \tag{3.16}$$

where

$$\mathcal{S}_{\ell,m} = \{u \in H^1(U_{\ell,m}) \;:\; u = 0 \text{ on } (0, m) \times \partial K_\ell \text{ and } \{m\} \times K_\ell\}. \tag{3.17}$$

The proof of Theorem 3.6 consists of showing that $t(\ell, m) \to d(\mathfrak{b}, \nu; \ell)$ and that a minimizer $\varphi_{\ell,m}$ of $\mathcal{G}_{\ell,m}$ converges to a minimizer of $\mathcal{G}_{\mathfrak{b},\nu;\ell}$ as $m \to \infty$. That will be done in several steps.

Step 1. In this step, we prove that $t(\ell, m) \to d(\mathfrak{b}, \nu; \ell)$ as $m \to \infty$, where $d(\mathfrak{b}, \nu; \ell)$ is the ground state energy introduced in (3.12). Recall the space $\mathcal{S}_\ell$ introduced in (3.10). Since every $u \in \mathcal{S}_{\ell,m}$ can be extended by 0 to a function $\widetilde{u} \in \mathcal{S}_\ell$, we get that, $t(\ell, m) \ge d(\mathfrak{b}, \nu; \ell)$. So, we need only prove that,

$$\limsup_{m \to \infty} t(\ell, m) \le d(\mathfrak{b}, \nu; \ell). \tag{3.18}$$

Let $(\varphi_n) \subset \mathcal{S}_\ell$ be a minimizing sequence of $\mathcal{G}_{\mathfrak{b},\nu;\ell}$, i.e.

$$d(\mathfrak{b}, \nu; \ell) = \lim_{n \to \infty} \mathcal{G}_{\mathfrak{b},\nu;\ell}(\varphi_n),$$

where $\mathcal{G}_{\mathfrak{b},\nu;\ell}(\varphi_n)$ is the functional introduced in (3.11).

Let $\eta \in C_c^\infty(\mathbb{R})$ be a cut-off function such that,

$$0 \le \eta \le 1 \text{ in } \mathbb{R}, \quad \eta = 1 \text{ in } [-1/2, 1/2], \quad \operatorname{supp} \eta \subset [-1, 1].$$

Let $\eta_m(x_1) = \eta(x_1/m)$, then $\eta_m(x_1)\varphi_n(x) \in \mathcal{S}_{\ell,m}$ and consequently,

$$t(\ell, m) \le \mathcal{G}_{\mathfrak{b},\nu;\ell}(\eta_m \varphi_n). \tag{3.19}$$

Using the inequality,

$$|(\nabla - i\mathbf{E}_\nu)(\eta_m\varphi_n)|^2 \le (1 + \varepsilon)|\eta_m(\nabla - i\mathbf{E}_\nu)\varphi_n|^2 + 2\varepsilon^{-1}|\nabla\eta_m|^2|\varphi_n|^2,$$

valid for all $\varepsilon \in (0, 1)$, together with the fact that $0 \le \eta_m \le 1$, we get the following estimate,

$$t(\ell, m) \le (1 + \varepsilon)\mathcal{G}_{\mathfrak{b},\nu;\ell}(\varphi_n) + \frac{2\varepsilon^{-1}}{m^2}\|\eta'\|^2_{L^\infty(\mathbb{R})} \int_{U_\ell} |\varphi_n|^2 \, dx + \mathfrak{b} \int_{U_\ell} \left(1 - \eta_m^2 + \varepsilon\right) |\varphi_n|^2 \, dx. \tag{3.20}$$

Taking $\limsup_{m\to\infty}$ on both sides of (3.20), we get (by using in particular Lebesgue's dominated convergence theorem),

$$\limsup_{m\to\infty} t(\ell,m) \le (1+\varepsilon)\mathcal{G}_{\mathfrak{b},\nu;\ell}(\varphi_n) + \mathfrak{b}\varepsilon\int_{U_\ell}|\varphi_n|^2\,dx\,.$$

Taking successively $\varepsilon\to 0_+$ then $n\to\infty$, we get the estimate in (3.18).

Step 2. Since $U_{\ell,m}$ is a bounded domain, it is obvious that $\mathcal{G}_{\ell,m}$ has a minimizer $\varphi_{\ell,m}$, see e.g. [8, Chapter 11]. The function $\varphi_{\ell,m}$ satisfies the following equation,

$$-(\nabla - i\mathbf{E}_\nu)^2\varphi_{\ell,m} = \mathfrak{b}(1-|\varphi_{\ell,m}|^2)\varphi_{\ell,m}\,,\quad \text{in } U_{\ell,m}\,. \tag{3.21}$$

A simple application of the maximum principle yields $|\varphi_{\ell,m}|\le 1$ everywhere. We will prove the following estimate,

$$\int_{U_{\ell,m}\cap\{x_1\ge 4\}}\frac{x_1}{(\ln x_1)^2}\left(|(\nabla - i\mathbf{E}_\nu)\varphi_{\ell,m}|^2 + |\varphi_{\ell,m}|^2 + x_1^2|\varphi_{\ell,m}|^4\right)dx \le C\ell^2\,, \tag{3.22}$$

valid for all $\ell>0$, where C is a universal constant.

We obtain the estimate (3.22) by following a construction similar to [21]. Consider a function $\chi\in C^\infty(\mathbb{R})$ such that $\operatorname{supp}\chi\subset(0,\infty)$. Using the equation in (3.21) and an integration by parts, we get,

$$\int_{U_{\ell,m}}\left(|(\nabla - i\mathbf{E}_\nu)\chi\varphi_{\ell,m}|^2 - \mathfrak{b}|\chi\varphi_{\ell,m}|^2 + \mathfrak{b}\chi^2|\varphi_{\ell,m}|^4\right)dx = \int_{U_{\ell,m}}|\chi'|^2|\varphi_{\ell,m}|^2\,dx\,. \tag{3.23}$$

Since $\chi\varphi_{\ell,m}\in H^1(\mathbb{R}^3)$ and the first eigenvalue of the Schrödinger operator with constant unit magnetic field in $L^2(\mathbb{R}^3)$ is equal to 1, we get the lower bound,

$$\int_{U_{\ell,m}}|(\nabla - i\mathbf{E}_\nu)\chi\varphi_{\ell,m}|^2\,dx \ge \int_{U_{\ell,m}}|\chi\varphi_{\ell,m}|^2\,dx\,.$$

Inserting this lower bound into (3.23) and remembering that $\mathfrak{b}\in[\Theta_0,1]$, we deduce the following estimate,

$$\int_{U_{\ell,m}}\chi^2|\varphi_{\ell,m}|^4\,dx \le \frac{1}{\mathfrak{b}}\int_{U_{\ell,m}}|\chi'|^2|\varphi_{\ell,m}|^2\,dx\,. \tag{3.24}$$

We select the function χ so that,

$$\chi(x_1)=0 \text{ if } x_1\le 0\,,\quad \chi(x_1)=\frac{x_1^{3/2}}{\ln x_1} \text{ if } x_1\ge 4\,. \tag{3.25}$$

The function χ consequently satisfies

$$0<\chi'(x_1)<\frac{3\sqrt{x_1}}{2\ln x_1}\quad\text{for all } x_1\ge 4\,. \tag{3.26}$$

Using the properties in (3.25) and (3.26) of the function χ, we infer from (3.24),

$$\begin{aligned}\int_{U_{\ell,m}\cap\{x_1\ge 4\}}&\frac{x_1^3}{(\ln x_1)^2}|\varphi_{\ell,m}|^4\,dx\\ &\le\frac{1}{\mathfrak{b}}\left[\frac{9}{4}\int_{U_{\ell,m}\cap\{x_1\ge 4\}}\frac{x_1}{(\ln x_1)^2}|\varphi_{\ell,m}|^2\,dx + \int_{U_{\ell,m}\cap\{x_1\le 4\}}|\chi'|^2|\varphi_{\ell,m}|^2\,dx\right]\\ &\le C\left(\int_{U_{\ell,m}\cap\{x_1\ge 4\}}\frac{x_1^3}{(\ln x_1)^2}|\varphi_{\ell,m}|^4\,dx\right)^{1/2}\ell + C\ell^2\,,\end{aligned} \tag{3.27}$$

where C is a universal constant. We explain how we get the estimate in (3.27). Actually, using that $|\varphi_{\ell,m}|\leq 1$, we get,

$$\int_{U_{\ell,m}\cap\{x_1\leq 4\}}|\chi'|^2|\varphi_{\ell,m}|^2\,dx\leq\|\chi'\|_{L^\infty([0,4])}(16\ell^2)\,.$$

Also, using a Cauchy-Schwarz inequality, we have,

$$\begin{aligned}\int_{U_{\ell,m}\cap\{x_1\geq 4\}}&\frac{x_1}{(\ln x_1)^2}|\varphi_{\ell,m}|^2\,dx\\&\leq\left(\int_{U_{\ell,m}\cap\{x_1\geq 4\}}\frac{1}{x_1(\ln x_1)^2}\,dx\right)^{1/2}\left(\int_{U_{\ell,m}\cap\{x_1\geq 4\}}\frac{x_1^3}{(\ln x_1)^2}|\varphi_{\ell,m}|^4\,dx\right)^{1/2},\end{aligned}$$

thereby yielding (3.27).

As a consequence of (3.27), we deduce the following estimate,

$$\forall\,\ell\geq 1\,,\quad\int_{U_{\ell,m}\cap\{x_1\geq 4\}}\frac{x_1}{(\ln x_1)^2}\left(|\varphi_{\ell,m}|^2+x_1^2|\varphi_{\ell,m}|^4\right)\,dx\leq C\ell^2\,,\tag{3.28}$$

for a possibly new universal constant C. Thus, to finish the proof of (3.22), we only need to prove that

$$\int_{U_{\ell,m}\cap\{x_1\geq 4\}}\frac{x_1}{(\ln x_1)^2}|(\nabla-i\mathbf{E}_\nu)\varphi_{\ell,m}|^2\,dx\leq C\ell^2\,.\tag{3.29}$$

To get (3.29), we select χ so that $\chi(x_1)=\sqrt{x_1}/\ln x_1$ if $x_1\geq 4$, and $\chi(x_1)=0$ if $x_1\leq 1$. Using the estimate,

$$\int_{U_{\ell,m}\cap\{x_1\geq 4\}}\chi^2|(\nabla-i\mathbf{E}_\nu)\varphi_{\ell,m}|^2\,dx\leq 2\int_{U_{\ell,m}}\left(|(\nabla-i\mathbf{E}_\nu)\chi\varphi_{\ell,m}|^2+|\chi'|^2|\varphi_{\ell,m}|^2\right)\,dx\,,$$

together with the estimates in (3.23), (3.28) and the properties of χ, we get easily the estimate in (3.29).

Step 3. In this step, we prove that there exists a minimizer φ_ℓ of $\mathcal{G}_{\mathfrak{b},\nu;\ell}$ satisfying the estimate in (3.14).

Starting from the bound $|\varphi_{\ell,m}|\leq 1$ and the equation (3.21), we get by a compactness and a diagonal sequence argument that there exists a function $\varphi_\ell\in C^1(U_\ell)$ such that, as $m\to\infty$,

$$\varphi_{\ell,m}\to\varphi_\ell\quad\text{in}\quad C^1(K)\,,$$

for any compact set $K\subset\overline{U_\ell}$. Furthermore, φ_ℓ satisfies the equation,

$$-(\nabla-i\mathbf{E}_\nu)^2\varphi_\ell=\mathfrak{b}(1-|\varphi_\ell|^2)\varphi_\ell\,,\quad\text{in }U_\ell\,,\tag{3.30}$$

together with the boundary conditions,

$$\frac{\partial\varphi_\ell}{\partial x_1}=0\text{ on }\{0\}\times K_\ell\,,\quad\varphi_\ell=0\text{ on }(0,\infty)\times\partial K_\ell\,.$$

Using the estimate in (3.22) together with the bound $|\varphi_{\ell,m}|\leq 1$, we get,

$$\int_{U_{\ell,m}}\left(|(\nabla-i\mathbf{E}_\nu)\varphi_{\ell,m}|^2+|\varphi_{\ell,m}|^2\right)\,dx\leq C\ell^2\,.$$

Passing to a subsequence, we conclude that, as $m\to\infty$, $\varphi_{\ell,m}\rightharpoonup\varphi_\ell$ and $(\nabla-i\mathbf{E}_\nu)\varphi_{\ell,m}\rightharpoonup(\nabla-i\mathbf{E}_\nu)\varphi_\ell$ weakly in $L^2(U_\ell)$. Consequently, we get that φ_ℓ is in the space $\mathcal{S}_\ell$,

$$\int_{U_{\ell,m}}|\varphi_{\ell,m}|^4\,dx\to\int_{U_\ell}|\varphi_\ell|^4\,dx\qquad\text{as }m\to\infty\,,$$

and φ_ℓ satisfies the estimate in (3.14).

Using the equation (3.21) and an integration by parts, it is easy to check that,

$$t(\ell,m)=-\frac{\mathfrak{b}}{2}\int_{U_{\ell,m}}|\varphi_{\ell,m}|^4\,dx\,.$$

Letting $m\to\infty$ and using Step 1 above, we get,

$$d(\mathfrak{b},\nu;\ell)=-\frac{\mathfrak{b}}{2}\int_{U_\ell}|\varphi_\ell|^4\,dx\,.$$

But, thanks to (3.30), an integration by parts yields,

$$\mathcal{G}_{\mathfrak{b},\nu;\ell}(\varphi_\ell)=-\frac{\mathfrak{b}}{2}\int_{U_\ell}|\varphi_\ell|^4\,dx\,,$$

thereby obtaining that φ_ℓ is a minimizer of $\mathcal{G}_{\mathfrak{b},\nu;\ell}$. □

3.3.2. *A rough estimate of the ground state energy.*

Lemma 3.7. *There exists a universal constant $C_1>0$ such that, for all $\mathfrak{b}\in[\Theta_0,1]$, $\nu\in[0,\pi/2]$ and $\ell>0$, we have,*

$$d(\mathfrak{b},\nu;\ell)\geq C_1\min(\zeta(\nu)-\mathfrak{b},0)\ell^2\,.$$

Furthermore, if $\nu\in[0,\pi/2)$ satisfies $\zeta(\nu)<\mathfrak{b}$, there exist positive constants C_2 and ℓ_0 such that, if $\ell\geq\ell_0$, then,

$$d(\mathfrak{b},\nu;\ell)\leq -C_2\ell^2$$

Remark 3.8. It results from Lemma 3.7 that

$$-\infty<\inf_{\ell\geq\ell_0}\frac{d(\mathfrak{b},\nu;\ell)}{\ell^2}\leq\sup_{\ell\geq\ell_0}\frac{d(\mathfrak{b},\nu;\ell)}{\ell^2}<0\,,$$

provided that $\mathfrak{b}$ and ν satisfy the conditions in Lemma 3.7.

Proof of Lemma 3.7.

Lower bound: Let $u_{\mathfrak{b}}\in\mathcal{S}_\ell$ be a minimizer of $\mathcal{G}_{\mathfrak{b},\nu;\ell}$. Since $u_{\mathfrak{b}}=0$ on $(0,\infty)\times\partial K_\ell$, we can extend $u_{\mathfrak{b}}$ to a function $\widetilde{u}_{\mathfrak{b}}$ by letting $\widetilde{u}_{\mathfrak{b}}=0$ in $(0,\infty)\times(\mathbb{R}^2\setminus K_\ell)$. The function $\widetilde{u}_{\mathfrak{b}}$ is in the form domain of the operator $\mathcal{L}(\nu)$ introduced in (3.5) and,

$$\begin{aligned}d(\mathfrak{b},\nu;\ell)&=\mathcal{G}_{\mathfrak{b},\nu;\ell}(u_{\mathfrak{b}})\\&=\int_{\mathbb{R}^3_+}\left(|(\nabla-i\mathbf{E}_\nu)\widetilde{u}_{\mathfrak{b}}|^2-\mathfrak{b}|\widetilde{u}_{\mathfrak{b}}|^2+\frac{\mathfrak{b}}{2}|\widetilde{u}_{\mathfrak{b}}|^4\right)dx\,.\end{aligned}$$

Using the variational min-max principle, we get,

$$d(\mathfrak{b},\nu;\ell)\geq(\zeta(\nu)-\mathfrak{b})\int_{\mathbb{R}^3}|\widetilde{u}_{\mathfrak{b}}|^2\,dx=(\zeta(\nu)-\mathfrak{b})\int_{U_\ell}|u_{\mathfrak{b}}|^2\,dx\,.$$

By Theorem 3.6, we have $\displaystyle\int_{U_\ell}|u_{\mathfrak{b}}|^2\,dx\leq C_1\ell^2$, where C_1 is a universal constant. If $\zeta(\nu)-\mathfrak{b}<0$, then

$$d(\mathfrak{b},\nu;\ell)\geq C_1(\zeta(\nu)-\mathfrak{b})\ell^2\,.$$

If $\zeta(\nu)-\mathfrak{b}\geq0$, then Lemma 3.4 tells us that $d(\mathfrak{b},\nu;\ell)=0$, thereby obtaining the lower bound announced in Lemma 3.7.

Upper bound: We construct a test-configuration f_ℓ as follows. Suppose $\nu\in(0,\pi/2)$. Let $\chi\in C_c^\infty(\mathbb{R})$ be a cut-off function such that $0\leq\chi\leq1$, $\chi=1$ in $[-1/2,1/2]$ and $\chi=0$ in $\mathbb{R}\setminus(-1,1)$. For all $v=(s,t)\in\mathbb{R}^2$, we set

$$\eta_\ell(v)=\chi_\ell(s)\chi_\ell(t),\quad \chi_\ell(x)=\chi\left(\frac{x}{\ell}\right)\quad(x\in\mathbb{R})\,. \tag{3.31}$$

Let ϕ be the eigenfunction introduced in Remark 3.3, and let $M>0$ be a given real number. For $j\in\mathbb{Z}$ we set $c_j=Mj$. The number of such c_j that belong to the interval $(-\ell,\ell)$ is given

by $N(\ell, M) = 2\{\ell/M\} + 1$, where $\{r\}$ denotes the largest integer strictly less than $r \in \mathbb{R}$. For $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, we define our trial state $f_\ell(x)$ to be

$$f_\ell(x) = \eta_\ell(x_2, x_3) \sum_{j=-\{\ell/M\}}^{\{\ell/M\}} \exp\big(-ic_j \sin(\nu)x_3\big)\phi(x_1, x_2 - c_j)\,.$$

We compute $\mathcal{G}_{\mathfrak{b},\nu;\ell}(tf_\ell)$, where $t > 0$ is a constant whose choice will be specified later. Actually,

$$\begin{aligned}
\mathcal{G}_{\mathfrak{b},\nu;\ell}(tf_\ell) = t^2 \int_{U_\ell} &\Big|\eta_\ell(x_2, x_3) \sum_{j=-\{\ell/M\}}^{\{\ell/M\}} \exp\big(-ic_j \sin(\nu)x_3\big)(\partial_{x_1}\phi)(x_1, x_2 - c_j)\Big|^2 \\
&+ \Big|\eta_\ell(x_2, x_3) \sum_{j=-\{\ell/M\}}^{\{\ell/M\}} \exp\big(-ic_j \sin(\nu)x_3\big)(\partial_{x_2}\phi)(x_1, x_2 - c_j) \\
&+ \partial_{x_2}(\eta_\ell)(x_2, x_3) \sum_{j=-\{\ell/M\}}^{\{\ell/M\}} \exp\big(-ic_j \sin(\nu)x_3\big)\phi(x_1, x_2 - c_j)\Big|^2 \\
+ \Big|\eta_\ell(x_2, x_3) &\sum_{j=-\{\ell/M\}}^{\{\ell/M\}} \exp\big(-ic_j \sin(\nu)x_3\big)\big(\cos(\nu)x_1 + \sin(\nu)(x_2 - c_j)\big)\phi(x_1, x_2 - c_j) \\
&+ \partial_{x_3}\big(\eta_\ell(x_2, x_3)\big) \sum_{j=-\{\ell/M\}}^{\{\ell/M\}} \exp\big(-ic_j \sin(\nu)x_3\big)\phi(x_1, x_2 - c_j)\Big|^2 \\
&- \mathfrak{b}\Big|\eta_\ell(x_2, x_3) \sum_{j=-\{\ell/M\}}^{\{\ell/M\}} \exp\big(-ic_j \sin(\nu)x_3\big)\phi(x_1, x_2 - c_j)\Big|^2 \\
&+ \frac{\mathfrak{b}}{2}t^2\Big|\eta_\ell(x_2, x_3) \sum_{j=-\{\ell/M\}}^{\{\ell/M\}} \exp\big(-ic_j \sin(\nu)x_3\big)\phi(x_1, x_2 - c_j)\Big|^4 dx\,. \qquad (3.32)
\end{aligned}$$

We define $\mathcal{G}_{\rm main}(tf_\ell)$ and $\mathcal{G}_{\rm rest}(tf_\ell)$ as follows,

$$\begin{aligned}
\mathcal{G}_{\rm main}(tf_\ell) = t^2 \int_{U_\ell} \big|\eta_\ell(x_2, x_3)\big|^2 \sum_{j=-\{\ell/M\}}^{\{\ell/M\}} &\Big[\big|(\partial_{x_1}\phi)(x_1, x_2 - c_j)\big|^2 + \big|(\partial_{x_2}\phi)(x_1, x_2 - c_j)\big|^2 \\
&+ \big|\big(\cos(\nu)x_1 + \sin(\nu)(x_2 - c_j)\big)\phi(x_1, x_2 - c_j)\big|^2 \\
&- \mathfrak{b}|\phi(x_1, x_2 - c_j)|^2 + \frac{\mathfrak{b}}{2}t^2|\phi(x_1, x_2 - c_j)|^4\Big]\, dx
\end{aligned}$$

and

$$\mathcal{G}_{\rm rest}(tf_\ell) = \mathcal{G}_{\mathfrak{b},\nu;\ell}(tf_\ell) - \mathcal{G}_{\rm main}(tf_\ell).$$

We will show that $\mathcal{G}_{\rm rest}(tf_\ell)$ is small compared to $\mathcal{G}_{\rm main}(tf_\ell)$. We start by obtaining an upper bound on $\mathcal{G}_{\rm main}(tf_\ell)$. Using that $\eta_\ell(x_2, x_3) = \chi_\ell(x_2)\chi_\ell(x_3)$ and $0 \le \chi_\ell(x_2) \le 1$ we get for a fixed

integer j

$$\begin{aligned}
t^2 \int_{U_\ell} &\big|\eta_\ell(x_2,x_3)\big|^2 \Big[\big|(\partial_{x_1}\phi)(x_1,x_2-c_j)\big|^2 + \big|(\partial_{x_2}\phi)(x_1,x_2-c_j)\big|^2 \\
&\quad + \big|\big(\cos(\nu)x_1+\sin(\nu)(x_2-c_j)\big)\phi(x_1,x_2-c_j)\big|^2 + \frac{\mathfrak{b}}{2}t^2|\phi(x_1,x_2-c_j)|^4\Big]\,dx \\
&\leq t^2 \int_{\mathbb{R}} |\chi_\ell(x_3)|^2\,dx_3 \int_{\mathbb{R}^2_+} \big|(\partial_{x_1}\phi)(x_1,x_2-c_j)\big|^2 + \big|(\partial_{x_2}\phi)(x_1,x_2-c_j)\big|^2 \\
&\quad + \big|\big(\cos(\nu)x_1+\sin(\nu)(x_2-c_j)\big)\phi(x_1,x_2-c_j)\big|^2 + \frac{\mathfrak{b}}{2}t^2|\phi(x_1,x_2-c_j)|^4\,dx_1dx_2 \\
&= \ell\lambda t^2\Big[\zeta(\nu) + \frac{\mathfrak{b}}{2}t^2\int_{\mathbb{R}^2_+}|\phi(x_1,x_2)|^4\,dx_1dx_2\Big]
\end{aligned}$$

where $\lambda = \int_{\mathbb{R}} |\chi(z)|^2\,dz$. For the negative term, we have

$$\begin{aligned}
-\mathfrak{b}t^2 \int_{U_\ell} |\eta_\ell(x_2,x_3)\phi(x_1,x_2-c_j)|^2\,dx &= -\mathfrak{b}t^2 \int_{\mathbb{R}^3_+} |\eta_\ell(x_2,x_3)\phi(x_1,x_2-c_j)|^2\,dx \\
&= -\mathfrak{b}t^2 \int_{\mathbb{R}} |\chi_\ell(x_3)|^2\,dx_3 \int_{\mathbb{R}^2_+} \big|\chi_\ell(x_2)\phi(x_1,x_2-c_j)\big|^2\,dx_1dx_2 \\
&= -\mathfrak{b}\ell\lambda t^2 + \mathfrak{b}\ell\lambda t^2 \int_{\mathbb{R}^2_+} (1-\big|\chi_\ell(x_2)\big|^2)|\phi(x_1,x_2-c_j)|^2\,dx_1dx_2
\end{aligned}$$

Using the exponential decay of ϕ we find that for all $s>0$

$$-\mathfrak{b}t^2 \int_{U_\ell} |\eta_\ell(x_2,x_3)\phi(x_1,x_2-c_j)|^2\,dx = -\mathfrak{b}\ell\lambda t^2 + o(1/\ell^s)$$

as $\ell\to\infty$. We conclude that

$$\mathcal{G}_{\mathrm{main}}(tf_\ell) \leq N(\ell,M)\ell\lambda t^2\Big[\zeta(\nu) - \mathfrak{b} + \frac{\mathfrak{b}}{2}t^2\int_{\mathbb{R}^2_+}|\phi(x_1,x_2)|^4\,dx_1dx_2\Big] + N(\ell,M)o(1/\ell^s)$$

as $\ell\to\infty$. We select t sufficiently small so that,

$$\zeta(\nu) - \mathfrak{b} + \frac{\mathfrak{b}t^2}{2}\int_{\mathbb{R}^2_+}|\phi|^4\,dx_1dx_2 \leq \frac{\zeta(\nu)-\mathfrak{b}}{2},$$

to get

$$\mathcal{G}_{\mathrm{main}}(tf_\ell) \leq \frac{1}{2}N(\ell,M)\ell\lambda t^2(\zeta(\nu)-\mathfrak{b}) + N(\ell,M)o(1/\ell^s) \leq \frac{1}{2M}\ell^2\lambda t^2(\zeta(\nu)-\mathfrak{b}) \tag{3.33}$$

as $\ell\to\infty$.

All terms in $\mathcal{G}_{\mathrm{rest}}(tf_\ell)$ can be taken care of in the same way, using (3.8) (except for the term with $\partial_{x_2}\eta_\ell(x_2,x_3)$ which is exponentially small due to the decay of ϕ). For this reason we only show how to handle the terms in $\mathcal{G}_{\mathrm{rest}}(tf_\ell)$ that comes from the first absolute value in (3.32). We meet terms like

$$t^2\int_{U_\ell} |\eta_\ell(x_2,x_3)|^2 e^{-ic_j\sin(\nu)x_3}(\partial_{x_1}\phi)(x_1,x_2-c_j)e^{ic_k\sin(\nu)x_3}(\partial_{x_1}\phi)(x_1,x_2-c_k)\,dx, \tag{3.34}$$

where $j\neq k$ are such that c_j and c_k both belong to $(-\ell,\ell)$. We assume that $j<k$ and estimate the absolute value of this integral by

$$t^2\lambda\ell\int_{\mathbb{R}^2_+} |(\partial_{x_1})\phi(x_1,x_2-c_j)(\partial_{x_1})\phi(x_1,x_2-c_k)|\,dx_1dx_2.$$

Next we decompose the integral into two terms, one where $x_2 < (c_j + c_k)/2$ and another one where $x_2 > (c_j + c_k)/2$. Using (3.8) we get

$$\begin{aligned}
&\int_{\substack{(x_1,x_2)\in\mathbb{R}^2_+\\ x_2<(c_j+c_k)/2}} |(\partial_{x_1})\phi(x_1,x_2-c_j)(\partial_{x_1})\phi(x_1,x_2-c_k)|\,dx_1dx_2\\
&= \int_{\substack{(x_1,x_2)\in\mathbb{R}^2_+\\ x_2<(c_j+c_k)/2}} e^{-\alpha/2\sqrt{x_1^2+(x_2-c_k)^2}}e^{\alpha/2\sqrt{x_1^2+(x_2-c_k)^2}}|(\partial_{x_1})\phi(x_1,x_2-c_j)(\partial_{x_1})\phi(x_1,x_2-c_k)|\,dx_1dx_2\\
&\qquad\qquad \leq e^{-\alpha M(k-j)/4}\|\partial_{x_1}\phi\|_{L^2(\mathbb{R}^2_+)}\|e^{\alpha/2\sqrt{x_1^2+(x_2)^2}}\partial_{x_1}\phi\|_{L^2(\mathbb{R}^2_+)}\\
&\qquad\qquad\qquad \leq \sqrt{\zeta(\nu)}\sqrt{C_{\nu,\alpha}}e^{-\alpha M(k-j)/4}.
\end{aligned}$$

Here we have used the L^∞ bound on the first exponential, and then the Cauchy-Schwarz inequality together with the exponential decay on ϕ from (3.8). The same bound is true for the integral where $x_2 > (c_j + c_k)/2$. Looking at the case $j > k$ we get the same bound, but with $j-k$ instead of $k-j$ in the exponential. We thus find that (3.34) in absolute value is bounded by

$$2t^2\lambda\ell\sqrt{\zeta(\nu)}\sqrt{C_{\nu,\alpha}}e^{-\frac{\alpha M}{4}|k-j|}.$$

We want to sum over all integers $j \neq k$ running from $-\{\ell/M\}$ to $\{\ell/M\}$. For a real number $a < 1$, the sum

$$2\sum_{j=1}^{n-1}(n-j)a^j = \frac{2a}{(1-a)^2}\big(a^n - 1 + n(1-a)\big) \leq \frac{2na}{(1-a)^2}.$$

Our sum transforms into this sum with $a = e^{-\alpha M/4}$ and $n = \{\ell/M\} \leq \ell/M$ we find that the error coming from the first absolute value in (3.32) is bounded in absolute value by

$$4t^2\lambda\ell^2\sqrt{\zeta(\nu)}\sqrt{C_{\nu,\alpha}}\frac{e^{-\alpha M/4}}{M(1-e^{-\alpha M/4})^2}.$$

Doing the same estimates for the other terms, we find that there exists a $\beta > 0$ and a constant $C > 0$ (not depending on M) such that

$$\mathcal{G}_{\rm rest}(tf_\ell) \leq Ct^2\lambda e^{-\beta M}\ell^2\,. \tag{3.35}$$

Combining (3.33) and (3.35) we find that

$$\mathcal{G}_{\mathfrak{b},\nu;\ell}(tf_\ell) \leq \Big[\frac{1}{2M}(\zeta(\nu)-\mathfrak{b}) + Ce^{-\beta M}\Big]\lambda t^2\ell^2.$$

Choosing M large enough we find that there exist ℓ_0 and $C_2 > 0$ such that if $\ell \geq \ell_0$ then $\mathcal{G}_{\mathfrak{b},\nu;\ell}(tf_\ell) \leq -C_2\ell^2$. This finishes the proof for $\nu \in (0,\pi/2)$.

Suppose that $\nu = 0$. In this specific case, we select the trial function f_ℓ as follows,

$$f_\ell(x) = \eta_\ell(x_2,x_3)\varphi_0(x_1)\,,$$

where φ_0 is the ground state introduced in (3.3), and η_ℓ is the cut-off function introduced in (3.31). Consider $t > 0$. An easy computation yields,

$$\mathcal{G}_{\mathfrak{b},0;\ell}(tf_\ell) \leq \lambda^2t^2\ell^2\left(\Theta_0 - \mathfrak{b} + \frac{\mathfrak{b}t^2}{2}\int_{\mathbb{R}_+}|\varphi_0(x_1)|^4\,dx_1\right) + Ct^2\,,$$

for some constant C. Selecting t sufficiently small so that $\Theta_0-\mathfrak{b}+\frac{\mathfrak{b}t^2}{2}\int_{\mathbb{R}_+}|\varphi_0(x_1)|^4\,dx_1 < \frac{\Theta_0-\mathfrak{b}}{2}$, we get the upper bound that we want to prove. □

3.3.3. *The thermodynamic limit.* The aim of this section is to prove the following theorem.

Theorem 3.9.

(1) *Suppose* $\mathfrak{b} \in [\Theta_0, 1]$ *and* $\nu \in [0, \pi/2]$. *There exists a constant* $E(\mathfrak{b}, \nu)$ *such that*

$$\liminf_{\ell\to\infty} \frac{d(\mathfrak{b}, \nu; \ell)}{4\ell^2} = \limsup_{\ell\to\infty} \frac{d(\mathfrak{b}, \nu; \ell)}{4\ell^2} = E(\mathfrak{b}, \nu)\,.$$

Furthermore, $E(\mathfrak{b}, \nu) < 0$ *if* $\mathfrak{b} > \zeta(\nu)$ *and* $E(\mathfrak{b}, \nu) = 0$ *otherwise.*

(2) *There exist positive universal constants* ℓ_0 *and* C *such that,*

$$E(\mathfrak{b}, \nu) \leq \frac{d(\mathfrak{b}, \nu; \ell)}{4\ell^2} \leq E(\mathfrak{b}, \nu) + \frac{C}{\ell^{2/3}}\,, \quad \forall\, \nu \in [0, \pi/2]\,, \quad \forall\, \ell \geq \ell_0\,. \tag{3.36}$$

The proof of Theorem 3.9 relies on the following abstract lemma, which is proved in [12].

Lemma 3.10. *Consider a decreasing function* $d : (0, \infty) \to (-\infty, 0]$ *such that the function* $f : (0, \infty) \ni \ell \mapsto \frac{d(\ell)}{\ell^2} \in \mathbb{R}$ *is bounded.*

Suppose that there exist constants $C > 0$ *and* $\ell_0 > 0$ *such that the estimate*

$$f(n\ell) \geq f((1 + a)\ell) - C\left(a + \frac{1}{a^2\ell^2}\right)\,, \tag{3.37}$$

holds true for all $a \in (0, 1)$, $n \in \mathbb{N}$ *and* $\ell \geq \ell_0$.

Then $f(\ell)$ *has a limit* A *as* $\ell \to \infty$. *Furthermore, for all* $\ell \geq 2\ell_0$, *the following estimate holds true,*

$$f(\ell) \leq A + \frac{2C}{\ell^{2/3}}\,. \tag{3.38}$$

In order to use the result of Lemma 3.10 for the energy $d(\mathfrak{b}, \nu, \ell)$, we establish the estimate in Lemma 3.11 below.

Lemma 3.11. *Let* $\mathfrak{b} \in (\Theta_0, 1)$ *and* $\nu \in (0, \pi/2)$ *such that* $\mathfrak{b} \geq \zeta(\nu)$. *There exist universal constants* $C > 0$ *and* $\ell_0 \geq 1$ *such that, for all* $\ell \geq \ell_0$, $n \in \mathbb{N}$ *and* $a \in (0, 1)$, *we have,*

$$\frac{d(\mathfrak{b}, \nu; n\ell)}{(n\ell)^2} \geq \frac{d(\mathfrak{b}, \nu; (1 + a)\ell)}{\ell^2} - \frac{C}{a^2\ell^2}\,.$$

Proof. Let $n \geq 2$ be a natural number. If $a \in (0, 1)$ and $j = (j_1, j_2) \in \mathbb{Z}^2$, we denote by

$$K_{a,j} = I_{j_1} \times I_{j_2}\,,$$

where

$$\forall\, m \in \mathbb{Z}\,, \quad I_m = \Big(m - n - 1 - a - (1 + a)\,,\, m - n - 1 - a + (1 + a)\Big)\,.$$

Consider a partition of unity (χ_j) of $\mathbb{R}^2$ such that:

$$\sum_j |\chi_j|^2 = 1\,, \quad 0 \leq \chi_j \leq 1 \quad \text{in } \mathbb{R}^2\,, \quad \operatorname{supp}\chi_j \subset K_{a,j}\,, \quad |\nabla\chi_j| \leq \frac{C}{a}\,,$$

where C is a universal constant. We define $\chi_{\ell,j}(x) = \chi_j(x/\ell)$. Then we obtain a new partition of unity $\chi_{\ell,j}$ such that $\operatorname{supp}\chi_{\ell,j} \subset \mathcal{K}_{\ell,j}$, with

$$\mathcal{K}_{\ell,j} = (\ell x \;:\; x \in K_{a,j}\}\,.$$

Let $\mathcal{J} = \{j = (j_1, j_2) \in \mathbb{Z}^2 \;:\; 1 \leq j_1, j_2 \leq n\}$ and $K_{n\ell} = (-n\ell, n\ell) \times (-n\ell, n\ell)$. Then the family $(\mathcal{K}_{\ell,j})_{j\in\mathcal{J}}$ is a covering of $K_{n\ell}$, and is formed exactly of n^2 squares.

We restrict the partition of unity $(\chi_{\ell,j})$ to the set $K_{n\ell} = (-n\ell, n\ell) \times (-n\ell, n\ell)$. Let $u_{n\ell}$ be a minimizer of (3.11), i.e. $\mathcal{G}_{\mathfrak{b},\nu;n\ell}(u_{n\ell}) = d(\mathfrak{b}, \nu; n\ell)$. We have the following decomposition formula,

$$\mathcal{G}_{\mathfrak{b},\nu;n\ell}(u_{n\ell}) \geq \sum_{j\in\mathcal{J}} \Big(\mathcal{G}_{\mathfrak{b},\nu;n\ell}(\chi_{\ell,j}u_{n\ell}) - \|\,|\nabla\chi_{\ell,j}|\,u_{n\ell}\|^2_{L^2(K_{n\ell})}\Big)\,. \tag{3.39}$$

By magnetic translation invariance, we get $\mathcal{G}_{\mathfrak{b},\nu;n\ell}(\chi_{\ell,j}u_{n\ell}) \geq d(\mathfrak{b},\nu;(1+a)\ell)$. Therefore, it results from (3.39),

$$d(\mathfrak{b},\nu;n\ell) \geq n^2 d(\mathfrak{b},\nu;(1+a)\ell) - \sum_j \|\nabla\chi_{\ell,j}\, u_{n\ell}\|^2_{L^2(K_{n\ell})}\,. \tag{3.40}$$

Using Theorem 3.6, we get further,

$$d(\mathfrak{b},\nu;n\ell) \geq n^2 d(\mathfrak{b},\nu;(1+a)\ell) - \frac{Cn^2}{a^2}\,. \tag{3.41}$$

Therefore,

$$\frac{d(\mathfrak{b},\nu;n\ell)}{(n\ell)^2} \geq \frac{d(\mathfrak{b},\nu;(1+a)\ell)}{\ell^2} - \frac{C}{a^2\ell^2}\,.$$

$\square$

Remark 3.12. Using the lower bound in Lemma 3.7, we infer from Lemma 3.11 the following lower bound,

$$\frac{d(\mathfrak{b},\nu;n\ell)}{(n\ell)^2} \geq \frac{d(\mathfrak{b},\nu;(1+a)\ell)}{((1+a)\ell)^2} - C\left(a + \frac{1}{a^2\ell^2}\right)\,,$$

where the constant $C > 0$ is universal.

Proof of Theorem 3.9.

Let $f(\ell) = \dfrac{d(\mathfrak{b},\nu;\ell)}{\ell^2}$. Thanks to Lemmas 3.7 and 3.11, we know that the functions $f(\ell)$ and $d(\ell) := d(\mathfrak{b},\nu;\ell)$ satisfy the assumptions in Lemma 3.10. Consequently, $f(\ell)$ has a limit $E(\mathfrak{b},\nu;\ell)$ as $\ell \to \infty$ and,

$$f(\ell) \leq E(\mathfrak{b},\nu) + \frac{C}{\ell^{2/3}}\,,$$

for all $\ell \geq \ell_0$. Here C and ℓ_0 are constants independent of $\mathfrak{b} \in [\Theta_0, 1]$ and $\nu \in [0, \pi/2]$.

It remains to establish a lower bound of $f(\ell)$. Let $n \in \mathbb{N}$. By using a comparison argument and magnetic translation invariance, we have that

$$d(\mathfrak{b},\nu;n\ell) \leq n^2 d(\mathfrak{b},\nu;\ell)\,.$$

Consequently,

$$\frac{d(\mathfrak{b},\nu;\ell)}{\ell^2} \geq \frac{d(\mathfrak{b},\nu;n\ell)}{(n\ell)^2}\,.$$

Making $n \to \infty$ in both sides above, we conclude,

$$\frac{d(\mathfrak{b},\nu;\ell)}{\ell^2} \geq E(\mathfrak{b},\nu)\,. \tag{3.42}$$

Lemma 3.7 tells us that if $\mathfrak{b} \geq \zeta(\nu)$, then $d(\mathfrak{b},\nu;\ell) < 0$. In this case, we get $E(\mathfrak{b},\nu) < 0$ as consequence of (3.42). On the other hand, If $\mathfrak{b} \leq \zeta(\nu)$, we know by Lemma 3.4 that $E(\mathfrak{b},\nu) = 0$. $\square$

3.3.4. *Properties of the function $E(\mathfrak{b},\nu)$.* Theorem 3.9 provides us with a limiting constant $E(\mathfrak{b},\nu) \in (-\infty, 0]$ defined for $\mathfrak{b} \in [\Theta_0, 1]$ and $\nu \in [0, \pi/2]$. In this section, we will study properties of $E(\mathfrak{b},\nu)$ as a function of $\mathfrak{b}$ and ν.

Theorem 3.13. *For all $(\mathfrak{b},\nu) \in [\Theta_0, 1] \times [0, \pi/2]$, let the constant $E(\mathfrak{b},\nu)$ be defined as in Theorem 3.9. Then $E(\mathfrak{b},\nu)$ satisfies the following properties.*

1. *Given $\nu \in [0, \pi/2]$, the function $[\Theta_0, 1] \ni \mathfrak{b} \mapsto E(\mathfrak{b},\nu)$ is continuous and monotone decreasing.*
2. *Given $\mathfrak{b} \in [\Theta_0, 1]$, the function $[0, \pi/2] \ni \nu \mapsto E(\mathfrak{b},\nu)$ is continuous.*

Proof. **Continuity and monotonicity of $\mathfrak{b}\mapsto E(\mathfrak{b},\nu)$:**

Let $\mathfrak{b}\in[\Theta_0,1]$ and $\varepsilon\in\mathbb{R}$ such that $\mathfrak{b}+\varepsilon\in[\Theta_0,1]$. Recall the definition of the functional $\mathcal{G}_{\mathfrak{b},\nu;\ell}$ in (3.11) together with the associated ground state energy $d(\mathfrak{b},\nu;\ell)$. It is easy to check that,

$$\mathcal{G}_{\mathfrak{b}+\varepsilon,\nu;\ell}(u)=\mathcal{G}_{\mathfrak{b},\nu;\ell}(u)+\varepsilon\int_{U_\ell}\left(-|u|^2+\frac{1}{2}|u|^4\right)dx\,, \tag{3.43}$$

valid for any $u\in\mathcal{S}_\ell$. In particular, setting successively $u=\varphi_{\mathfrak{b},\nu;\ell}$ then $u=\varphi_{\mathfrak{b}+\varepsilon,\nu;\ell}$ in (3.43) and using the properties in Theorem 3.6, we get the following estimate,

$$|d(\mathfrak{b}+\varepsilon,\nu;\ell)-d(\mathfrak{b},\nu;\ell)|\le C\varepsilon\ell^2\,,$$

for some universal constant C. Remembering the definition of $E(\cdot,\nu)$, we get,

$$|E(\mathfrak{b}+\varepsilon,\nu)-E(\mathfrak{b},\nu)|\le C\varepsilon\,,$$

thereby proving the continuity of $E(\cdot,\nu)$.

To obtain monotonicity of $E(\mathfrak{b},\nu)$, we suppose that $\varepsilon<0$ and we set $u=\varphi_{\mathfrak{b}+\varepsilon,\nu;\ell}$ in (3.43). Thanks to Theorem 3.6, we know that $|u|\le 1$ and consequently,

$$\begin{aligned}d(\mathfrak{b}+\varepsilon,\nu;\ell)&\ge d(\mathfrak{b},\nu;\ell)-\frac{\varepsilon}{2}\int_{U_\ell}|u|^2\,dx\\&\ge d(\mathfrak{b},\nu;\ell)\,.\end{aligned}$$

Dividing both sides above by $4\ell^2$ then letting $\ell\to\infty$, we get,

$$E(\mathfrak{b}+\varepsilon,\nu;\ell)\ge E(\mathfrak{b},\nu;\ell)\,,$$

which proves that $E(\cdot,\nu)$ is monotone decreasing.

Continuity of $\nu\mapsto E(\mathfrak{b},\nu)$:

Let $\nu\in[0,\pi/2]$ and $\varepsilon\in(-1,1)\setminus\{0\}$ such that $\nu+\varepsilon\in[0,\pi/2]$. We want to prove that $E(\mathfrak{b},\nu+\varepsilon)\to E(\mathfrak{b},\nu)$ as $\varepsilon\to 0$.

Recall the definition of the magnetic potential $\mathbf{E}_\nu$ in (3.4). Notice that, if $u\in\mathcal{S}_\ell$, we have the following estimate,

$$|\mathcal{G}_{\mathfrak{b},\nu+\varepsilon;\ell}(u)-\mathcal{G}_{\mathfrak{b},\nu;\ell}(u)|\le|\varepsilon|\int_{U_\ell}|(\nabla-i\mathbf{E}_\nu)u|^2\,dx+2|\varepsilon^{-1}|\int_{U_\ell}|(\mathbf{E}_{\nu+\varepsilon}-\mathbf{E}_\nu)u|^2\,dx\,. \tag{3.44}$$

Using the bounds,

$$|\cos(\nu+\varepsilon)-\cos\nu|\le\varepsilon\,,\quad|\sin(\nu+\varepsilon)-\sin\nu|\le\varepsilon\,,$$

we get,

$$|\mathbf{E}_{\nu+\varepsilon}(x)-\mathbf{E}_\nu(x)|\le\varepsilon\left(|x_1|+|x_2|\right),\quad\forall\,x=(x_1,x_2,x_3)\in\mathbb{R}^3\,.$$

Consequently, we infer from (3.44) the following bound,

$$|\mathcal{G}_{\mathfrak{b},\nu+\varepsilon;\ell}(u)-\mathcal{G}_{\mathfrak{b},\nu;\ell}(u)|\le|\varepsilon|\int_{U_\ell}|(\nabla-i\mathbf{E}_\nu)u|^2\,dx+2|\varepsilon|\int_{U_\ell}\left(|x_1|^2+|x_2|^2\right)|u|^2\,dx\,. \tag{3.45}$$

Let $\zeta\in C_c^\infty(\mathbb{R})$ be a cut-off function satisfying $0\le\zeta\le 1$ in $\mathbb{R}$, $\operatorname{supp}\zeta\subset[-1,1]$ and $\zeta=1$ in $[-\frac{1}{2},\frac{1}{2}]$. Let $\zeta_\ell(x)=\zeta(x_1/\ell)$ and $\varphi_{\mathfrak{b},\nu+\varepsilon;\ell}\in\mathcal{S}_\ell$ be a minimizer of $\mathcal{G}_{\mathfrak{b},\nu+\varepsilon;\ell}$ given by Theorem 3.6. Applying (3.45) with $u=\zeta_\ell\varphi_\ell$ and using the decay estimate of Theorem 3.6, we get for all $\ell\ge 1$,

$$\mathcal{G}_{\mathfrak{b},\nu+\varepsilon;\ell}(\zeta_\ell\varphi_{\mathfrak{b},\nu+\varepsilon;\ell})\ge d(\mathfrak{b},\nu;\ell)-C|\varepsilon|\ell^4\,, \tag{3.46}$$

where $C>0$ is a universal constant. Using the estimate in (3.36), we get for all $\ell\ge\ell_0$,

$$\mathcal{G}_{\mathfrak{b},\nu+\varepsilon;\ell}(\zeta_\ell\varphi_{\mathfrak{b},\nu+\varepsilon;\ell})\ge(4\ell^2)E(\mathfrak{b},\nu)-C|\varepsilon|\ell^4\,, \tag{3.47}$$

where ℓ_0 is a universal constant.

We estimate the term $\mathcal{G}_{\mathfrak{b},\nu+\varepsilon;\ell}(\zeta_\ell\varphi_{\mathfrak{b},\nu+\varepsilon;\ell})$ from above. Actually, an integration by parts give us the following identity,

$$\mathcal{G}_{\mathfrak{b},\nu+\varepsilon;\ell}(\zeta_\ell\varphi_{\mathfrak{b},\nu+\varepsilon;\ell})=\int_{U_\ell}|\zeta_\ell'|^2|\varphi_{\mathfrak{b},\nu+\varepsilon;\ell}|^2\,dx+\int_{U_\ell}\zeta_\ell^2\left(\frac{1}{2}\zeta_\ell^2-1\right)|\varphi_{\mathfrak{b},\nu+\varepsilon;\ell}|^4\,dx\,.$$

Using the decay estimate of Theorem 3.6, we get,

$$\mathcal{G}_{\mathfrak{b},\nu+\varepsilon;\ell}(\zeta_\ell\varphi_{\mathfrak{b},\nu+\varepsilon;\ell}) \leq C - \frac{1}{2}\int_{U_\ell} |\varphi_{\mathfrak{b},\nu+\varepsilon;\ell}|^4\,dx\,.$$

The Ginzburg-Landau equation satisfied by $\varphi_{\mathfrak{b},\nu+\varepsilon;\ell}$ gives us

$$\mathcal{G}_{\mathfrak{b},\nu+\varepsilon;\ell}(\varphi_{\mathfrak{b},\nu+\varepsilon;\ell}) = -\frac{1}{2}\int_{U_\ell} |\varphi_{\mathfrak{b},\nu+\varepsilon;\ell}|^4\,dx.$$

Consequently, we get the upper bound,

$$\mathcal{G}_{\mathfrak{b},\nu+\varepsilon;\ell}(\zeta_\ell\varphi_{\mathfrak{b},\nu+\varepsilon;\ell}) \leq d(\mathfrak{b},\nu+\varepsilon;\ell) + C\,.$$

Using the estimate in (3.36), we get further,

$$\mathcal{G}_{\mathfrak{b},\nu+\varepsilon;\ell}(\zeta_\ell\varphi_{\mathfrak{b},\nu+\varepsilon;\ell}) \leq (4\ell^2)E(\mathfrak{b},\nu+\varepsilon) + C\ell^{4/3}\,.$$

Inserting this upper bound into (3.47), we get,

$$E(\mathfrak{b},\nu+\varepsilon) \geq E(\mathfrak{b},\nu) - C|\varepsilon|\ell^4 - C\ell^{-2/3}\,.$$

Taking successively $\liminf\limits_{\varepsilon\to 0}$ then $\lim\limits_{\ell\to\infty}$ on both sides above, we get,

$$\liminf_{\varepsilon\to 0} E(\mathfrak{b},\nu+\varepsilon) \geq E(\mathfrak{b},\nu)\,.$$

In a similar fashion, by applying (3.44) with $u = \zeta_\ell\varphi_{\mathfrak{b},\nu;\ell}$ and $\varphi_{\mathfrak{b},\nu;\ell}$ being a minimizer of $\mathcal{G}_{\mathfrak{b},\nu;\ell}$, we can prove that $\limsup\limits_{\varepsilon\to 0} E(\mathfrak{b},\nu+\varepsilon) \leq E(\mathfrak{b},\nu)$. This gives us that

$$\lim_{\varepsilon\to 0} E(\mathfrak{b},\nu+\varepsilon) = E(\mathfrak{b},\nu)\,,$$

and thereby proves the continuity of the function $E(\mathfrak{b},\cdot)$. □

4. Auxiliary results

We collect some results that are necessary for controlling the errors resulting from various approximations. The first result is an estimate obtained in [4, Lemma 3.2 and Theorem 3.3] (for a different method see [8, Theorem 12.3.1]).

Lemma 4.1. *There exists a constant $C_1 > 0$ such that if $(\psi,\mathbf{A}) \in H^1(\Omega)\times \dot{H}^1_{\mathrm{div},\mathbf{F}}(\mathbb{R}^3)$ is a solution to* (1.3)*, then*

$$\|\psi\|^4_{L^4(\Omega)} \leq C_1\lambda.$$

Here,

$$\lambda = \max\left(\frac{1}{\kappa}, \left[\frac{\kappa}{H} - 1\right]^2_+\right)\,. \tag{4.1}$$

The next lemma is taken from [8, Lemma 10.33], which, together with Lemma 4.1, give a good estimate of $\|\operatorname{curl}(\mathbf{A}-\mathbf{F})\|_{L^2(\mathbb{R}^3)}$.

Lemma 4.2. *There exists a constant $C > 0$ such that, if $(\psi,\mathbf{A}) \in H^1(\Omega;\mathbb{C})\times \dot{H}^1_{\mathrm{div},\mathbf{F}}(\mathbb{R}^3)$ is a solution of* (1.3)*, then,*

$$\|\operatorname{curl}(\mathbf{A}-\mathbf{F})\|_{L^2(\mathbb{R}^3)} \leq \frac{C}{H}\|\psi\|^2_{L^4(\Omega)}\,, \tag{4.2}$$

for all $\kappa > 0$ and $H > 0$.

The next result is Theorem 4.3 below which is proved in [12]. Similar estimates to those in Theorem 4.3 are also given in [20].

Theorem 4.3. *Suppose that $0 < \Lambda_{\min} \leq \Lambda_{\max}$. There exist constants $\kappa_0 > 1$ and $C_1 > 0$ such that, if*

$$\kappa \geq \kappa_0\,, \quad \Lambda_{\min} \leq \frac{\kappa}{H} \leq \Lambda_{\max}\,,$$

and $(\psi, \mathbf{A}) \in H^1(\Omega;\mathbb{C}) \times \dot{H}^1_{\mathrm{div},\mathbf{F}}(\mathbb{R}^3)$ is a solution of (1.3), then

$$\|(\nabla - i\kappa H\mathbf{A})\psi\|_{C(\overline{\Omega})} \leq C_1\sqrt{\kappa H}\|\psi\|_{L^\infty(\Omega)}\,, \tag{4.3}$$

$$\|\mathbf{A}-\mathbf{F}\|_{W^{2,6}(\Omega)} \leq C_1\left(\|\operatorname{curl}(\mathbf{A}-\mathbf{F})\|_{L^2(\mathbb{R}^3)} + \frac{1}{\sqrt{\kappa H}}\|\psi\|_{L^6(\Omega)}\|\psi\|_{L^\infty(\Omega)}\right), \tag{4.4}$$

$$\|\mathbf{A}-\mathbf{F}\|_{C^{1,1/2}(\Omega)} \leq C_1\left(\|\operatorname{curl}(\mathbf{A}-\mathbf{F})\|_{L^2(\mathbb{R}^3)} + \frac{1}{\sqrt{\kappa H}}\|\psi\|_{L^6(\Omega)}\|\psi\|_{L^\infty(\Omega)}\right), \tag{4.5}$$

$$\|\operatorname{curl}(\mathbf{A}-\mathbf{F})\|_{C^{0,1/2}(\overline{\Omega})} \leq C_1\left(\|\operatorname{curl}(\mathbf{A}-\mathbf{F})\|_{L^2(\mathbb{R}^3)} + \frac{1}{\sqrt{\kappa H}}\|\psi\|_{L^6(\Omega)}\|\psi\|_{L^\infty(\Omega)}\right). \tag{4.6}$$

Combining the results in Lemma 4.1, Lemma 4.2 and Theorem 4.3, we get the following corollary.

Corollary 4.4. *Suppose that $0 < \Lambda_{\min} \leq \Lambda_{\max}$. There exist constants $\kappa_0 > 1$ and $C_1 > 0$ such that, if*

$$\kappa \geq \kappa_0\,, \quad \Lambda_{\min} \leq \frac{\kappa}{H} \leq \Lambda_{\max}\,,$$

and $(\psi, \mathbf{A}) \in H^1(\Omega;\mathbb{C}) \times \dot{H}^1_{\mathrm{div},\mathbf{F}}(\mathbb{R}^3)$ is a solution of (1.3), then,

$$\|\mathbf{A}-\mathbf{F}\|_{C^{1,1/2}(\Omega)} \leq C_1\frac{\lambda^{1/6}}{\kappa}\,.$$

5. Lower bound

In this section we will prove Theorem 5.1 below, whose statement requires some notation. Let $D \subset \Omega$ be a given open set such that there exists a subset $\widetilde{D}$ of $\mathbb{R}^3$ having smooth boundary and $D = \widetilde{D} \cap \Omega$. For all $a > 0$, we assign to D the following subset of Ω,

$$D_a = \{x \in \Omega \;:\; \operatorname{dist}(x, D) \leq a\}\,. \tag{5.1}$$

We introduce the following functional,

$$\mathcal{E}_0(u, \mathbf{A}; D) = \int_D \left(|(\nabla - i\mathbf{A})u|^2 - \kappa^2|u|^2 + \frac{\kappa^2}{2}|u|^4\right)\,dx\,, \tag{5.2}$$

where $u \in H^1(\Omega;\mathbb{C})$ and $\mathbf{A} \in \dot{H}^1_{\mathrm{div},\mathbf{F}}(\mathbb{R}^3)$. If $D = \Omega$, we omit the dependence on the domain and write $\mathcal{E}_0(\psi, \mathbf{A})$ for $\mathcal{E}_0(\psi, \mathbf{A}; \Omega)$.

Theorem 5.1. *Suppose that the magnetic field H is a function of κ such that,*

$$1 \leq \liminf_{\kappa\to\infty}\frac{H}{\kappa} \leq \limsup_{\kappa\to\infty}\frac{H}{\kappa} < \infty\,.$$

Let $\kappa \ni \mathbb{R}_+ \mapsto a(\kappa) \in \mathbb{R}_+$ be a function satisfying $\lim_{\kappa\to\infty} a(\kappa) = 0$. Then, for any solution $(\psi, \mathbf{A}) \in H^1(\Omega;\mathbb{C}) \times \dot{H}^1_{\mathrm{div},\mathbf{F}}(\mathbb{R}^3)$ of (1.1) and any function $h \in C^1(\Omega)$ satisfying $\|h\|_{L^\infty(\Omega)} \leq 1$ and $\operatorname{supp} h \subset \overline{D_{a(\kappa)}}$, the following asymptotic lower bound holds,

$$\mathcal{E}_0(h\psi, \mathbf{A}) \geq \sqrt{\kappa H}\int_{\overline{D}\cap\partial\Omega} E\left(\frac{\kappa}{H}, \nu(x)\right)\,d\sigma(x) + E_2|D|\,[\kappa - H]_+^2 + o\Big(\max\left(\kappa, [\kappa-H]_+^2\right)\Big) \quad \text{as} \quad \kappa\to\infty\,. \tag{5.3}$$

Here $d\sigma(x)$ is the surface measure on the boundary of Ω, $E_2 < 0$ is the universal constant introduced in (2.4), and $\mathcal{E}_0$ is the functional introduced in (5.2).

The proof of Theorem 5.1 is a bit lengthy and is divided into several subsections. We will introduce two parameters

$$\alpha = \alpha(\kappa) \in (0,1)\,, \quad \delta = \delta(\kappa) \in (0,1)\,,$$

that will be used along the proof. Different conditions on these parameters will arise so as to control the error terms correctly. The choice of the parameters will be fixed at the end of the proof (in (5.18) below) so that they satisfy the aforementioned conditions, and are negligible in the limit of large κ.

Throughout the section, $(\psi, \mathbf{A}) \in H^1(\Omega) \times \dot{H}^1_{\mathrm{div},\mathbf{F}}(\mathbb{R}^3)$ will always denote a solution to (1.3).

5.1. **Splitting into bulk and surface terms.** We introduce smooth real-valued functions χ_1 and χ_2 such that $\chi_1^2 + \chi_2^2 = 1$ in Ω,

$$\chi_1(x) = \begin{cases} 1, & \text{if } \mathrm{dist}(x, \partial\Omega) < \delta/2, \\ 0, & \text{if } \mathrm{dist}(x, \partial\Omega) > \delta, \end{cases}$$

and $|\nabla \chi_j| \le C/\delta$ for $j = 1, 2$ and some constant C independent of δ. Using the IMS decomposition formula and the fact that $\int_\Omega (\chi_j(x)^2 - \chi_j(x)^4)|\psi|^4\,dx \ge 0$ (since $0 \le \chi_j(x) \le 1$), we find that

$$\mathcal{E}_0(h\psi, \mathbf{A}) \ge \mathcal{E}_0(\chi_1 h\psi, \mathbf{A}) + \mathcal{E}_0(\chi_2 h\psi, \mathbf{A}) - \sum_{j=1}^{2} \int_\Omega |\nabla \chi_j|^2 |h\psi|^2\,dx. \tag{5.4}$$

To estimate the error we use that $\|\psi\|_\infty \le 1$ and the fact that the measure of the support of $\nabla\chi_j$ is bounded by $C\delta$ for some constant C, so get

$$\sum_{j=1}^{2} \int_\Omega |\nabla \chi_j|^2 |h\psi|^2\,dx \le C\delta^{-1}\,. \tag{5.5}$$

Next, we estimate separately the terms $\mathcal{E}_0(\chi_1 h\psi, \mathbf{A})$ (surface energy) and $\mathcal{E}_0(\chi_2 h\psi, \mathbf{A})$ (bulk energy).

5.2. **The surface energy.** The estimate of the surface energy requires two steps, a decomposition of the energy via a partition of unity, then passing to local boundary coordinates that allow us to compare with the model case of the half-space that is studied in Section 3.

5.2.1. *Boundary coordinates.* We introduce a system of coordinates valid near a point of the boundary. These coordinates are used in [16] and then in [22] in order to estimate the ground state energy of a magnetic Schrödinger operator with large magnetic field (or with small semi-classical parameter).

Consider a point $x_0 \in \partial\Omega$. After performing a translation, we may assume that the Cartesian coordinates of x_0 are all 0, i.e. $x_0 = 0$. Let V be a neighborhood of x_0 such that there exists local boundary coordinates (y_2, y_3) in $W = V \cap \partial\Omega$, i.e. there exists an open subset U of $\mathbb{R}^2$ and a diffeomorphism $\phi : W \to U$, $\phi(x) = (y_2, y_3)$. Denote by N the inward pointing normal at the point $\phi^{-1}(y_2, y_3) \in \partial\Omega$. We define the coordinate transformation Φ as

$$(x_1, x_2, x_3) = \Phi^{-1}(y_1, y_2, y_3) = \phi^{-1}(y_2, y_3) + y_1 N.$$

The standard Euclidean metric $g_0 = \sum_{j=1}^3 dx_j \otimes dx_j$ transforms to

$$\begin{aligned} g_0 &= \sum_{1\le j,k\le 3} g_{jk} dy_j \otimes dy_k \\ &= dy_3 \otimes dy_3 + \sum_{2\le j,k\le 3} \Big[G_{jk}(y_2,y_3) - 2y_1 K_{jk}(y_2,y_3) + y_1^2 L_{jk}(y_2,y_3)\Big] dy_j \otimes dy_k \end{aligned}$$

where

$$G=\sum_{2\le k,j\le 3}G_{jk}\,dy_j\otimes dy_k=\sum_{\substack{2\le k,j\le 3\\ 1\le l\le 3}}\Big\langle\frac{\partial x_l}{\partial y_j},\frac{\partial x_l}{\partial y_k}\Big\rangle\,dy_j\otimes dy_k\,,$$

$$K=\sum_{2\le k,j\le 3}K_{jk}\,dy_j\otimes dy_k=\sum_{2\le k,j\le 3}-\Big\langle\frac{\partial N}{\partial y_j},\frac{\partial x}{\partial y_k}\Big\rangle\,dy_j\otimes dy_k \text{ and}$$

$$L=\sum_{2\le k,j\le 3}L_{jk}\,dy_j\otimes dy_k=\sum_{2\le k,j\le 3}\Big\langle\frac{\partial N}{\partial y_j},\frac{\partial N}{\partial y_k}\Big\rangle\,dy_j\otimes dy_k$$

are the first, second and third fundamental forms on $\partial\Omega$. We denote by g^{jk} the inverse of g_{jk}. Its Taylor expansion, valid in the neighborhood, is given by

$$(g^{jk})_{1\le j,k\le 3}=Id+\begin{pmatrix}0&0&0\\0&\mathcal{O}(|y|)&\mathcal{O}(|y|)\\0&\mathcal{O}(|y|)&\mathcal{O}(|y|)\end{pmatrix}. \tag{5.6}$$

The Lebesgue measure dx transforms into $dx=\det(g_{jk})^{1/2}dy$. This determinant has the Taylor expansion

$$\det(g_{jk})^{1/2}=1+\mathcal{O}(|y|), \tag{5.7}$$

valid in the neighborhood. The magnetic vector potential $\mathbf{F}=(F_1,F_2,F_3)=(-x_2/2,x_1/2,0)$ is transformed to a magnetic potential $\widetilde{\mathbf{F}}=(\widetilde{F}_1,\widetilde{F}_2,\widetilde{F}_3)$ given as follows,

$$\widetilde{F}_j(y)=\sum_{k=1}^{3}F_k(\Phi^{-1}(y))\frac{\partial x_k}{\partial y_j}. \tag{5.8}$$

In [22], a particular choice of a gauge transformation is selected so that, in a neighborhood of the point x_0, the new vector potential $\widetilde{\mathbf{F}}$ satisfies,

$$\widetilde{F}_1=0,\qquad \widetilde{F}_2=\mathcal{O}(|y|^2),\qquad \widetilde{F}_3=y_1\cos\nu+y_2\sin\nu+\mathcal{O}(|y|^2), \tag{5.9}$$

Here $\nu=\nu(x_0)$ is the angle between the magnetic field and the tangent plane of $\partial\Omega$ at the point x_0.

Notice that the constants implicit in the $\mathcal{O}$ notation in (5.6), (5.7) and (5.9) can be chosen uniform (i.e. independent of the boundary point x_0) by compactness and regularity of $\partial\Omega$.

If u is a function with support in a coordinate neighborhood V, we may express the functional $\mathcal{E}_0(u,\mathbf{F})$ explicitly in the new coordinates as follows,

$$\mathcal{E}_0^{3\mathrm{D}}(u,\mathbf{F})=\int_{\Phi(V)}\det(g_{jk})^{1/2}\Big[\sum_{1\le j,k\le 3}g^{jk}\big(\partial_{y_j}-i\kappa H\widetilde{F}_j\big)\widetilde{u}\times\overline{\big(\partial_{y_k}-i\kappa H\widetilde{F}_k\big)\widetilde{u}}$$
$$-\kappa^2|\widetilde{u}|^2+\frac{\kappa^2}{2}|\widetilde{u}|^4\Big]\,dy, \tag{5.10}$$

where $\widetilde{u}=e^{i\kappa H\phi}u\circ\Phi^{-1}$ (with ϕ the gauge transformation necessary to pass to the $\widetilde{\mathbf{F}}$ given in (5.9)).

5.3. **Decomposition of the energy.** Consider a family $\{x_{0,l}\}_l$ of points on the boundary $\partial\Omega$ whose choice will be specified below. For each point $x_{0,l}$, we may introduce a coordinate transformation Φ_l valid near the point $x_{0,l}$. Actually, after performing a translation, we may reduce to the case corresponding to the coordinate transformation Φ valid near the point $x_0\in\partial\Omega$ as above.

We introduce a new partition of unity $\{\widetilde{\chi}_l\}_l$ covering the set $\Omega_1:=\operatorname{supp}\chi_1$, whose support will be centered at coordinate neighborhoods of a family of points $\{x_{0,l}\}$. Let $\alpha=\alpha(\kappa)\ll 1$ be

a parameter that will be explicitly chosen below. Let, for $\delta > 0$,

$$O_\delta := \big\{(y_1, y_2, y_3) \;\big|\; 0 < y_1,\ -\delta < y_2 < \delta,\ -\delta < y_3 < \delta\big\}\,.$$

We choose $\{\widetilde{\chi}_l\}_l$ as smooth non-negative functions such that $\sum_l \widetilde{\chi}_l^2(x) \equiv 1$ in Ω_1, $\widetilde{\chi}_l \equiv 1$ in the set $\Omega_1 \cap \Phi_l^{-1}\big(O_{(1-\alpha)\delta}\big)$, and $\operatorname{supp}\widetilde{\chi}_l \subset \Omega_1 \cap \Phi_l^{-1}\big(O_\delta\big)$, and such that there exists a constant $C > 0$ so that

$$\sum_l |\nabla \widetilde{\chi}_l(x)|^2 \leq C(\alpha\delta)^{-2}$$

and

$$\sum \widetilde{\chi}_l(x)^2 \leq C \qquad \text{(finite overlap)}$$

for all $x \in \Omega_1$.

Using the IMS decomposition formula and the inequality $\int_\Omega (\widetilde{\chi}_l(x)^2 - \widetilde{\chi}_l(x)^4)|\chi_1\psi|^4\,dx \geq 0$, we get the following lower bound of the surface energy,

$$\mathcal{E}_0^{\rm 3D}(\chi_1 h\psi, \mathbf{A}) \geq \sum_l \Big\{\mathcal{E}_0^{\rm 3D}(\widetilde{\chi}_l h\chi_1\psi, \mathbf{A}) - \int_{\Omega_1} |\nabla\widetilde{\chi}_l|^2 |\chi_1 h\psi|^2\,dx\Big\}. \tag{5.11}$$

Using the bound on $\nabla\widetilde{\chi}_l$, we bound the error term by

$$\sum_l \int_{\Omega_1} |\nabla\widetilde{\chi}_l|^2|\chi_1 h\psi|^2\,dx \leq C\alpha^{-2}\delta^{-1}\,.$$

We approximate the magnetic potential $\mathbf{A}$ by the magnetic potential $\mathbf{F}$. Part of the approximation relies on the construction of a suitable gauge transformation. Let

$$\phi_l(x) = \big(\mathbf{A}(x_{0,l}) - \mathbf{F}(x_{0,l})\big)\cdot x\,,$$

and

$$u_l(x) = e^{i\phi_l(x)}\,\widetilde{\chi}_l(x)\chi_1(x)h(x)\psi(x)\,.$$

Then it holds true that,

$$\mathcal{E}_0^{\rm 3D}(\widetilde{\chi}_l\chi_1 h\psi, \mathbf{A}) = \mathcal{E}_0^{\rm 3D}(u_l, \mathbf{A} - \nabla\phi_l)\,. \tag{5.12}$$

We will prove the following estimate (for large values of κ),

$$\mathcal{E}_0^{\rm 3D}(u_l, \mathbf{A} - \nabla\phi_l) \geq (1-\delta)\mathcal{E}_0^{\rm 3D}(u_l, \mathbf{F}) - C\delta\kappa^2\int_\Omega |u_l|^2\,dx\,. \tag{5.13}$$

The proof of (5.13) will be postponed to the end of the section. We will bound the energy $\mathcal{E}_0(u_l, \mathbf{F})$ from below by expressing it in boundary coordinates. Let $\nu = \nu_l = \nu(x_{0,l})$ be the angle between the applied magnetic field $\beta = (0, 0, 1)$ and the tangent plane of $\partial\Omega$ at the point $x_{0,l}$. Selecting the gauge giving us the magnetic potential $\widetilde{\mathbf{F}}$ in (5.9), we get with

$$\psi_l = u_l \circ \Phi_l \times \text{(a gauge transformation)},$$

that

$$\mathcal{E}_0^{\rm 3D}(u_l, \mathbf{F}) = \int_{O_\delta} \det(g_{jk})^{1/2}\Big[\sum_{1\leq j,k\leq 3} g^{jk}\big(\partial_{y_j} - i\kappa H\widetilde{F}_j\big)\psi_l \times \overline{\big(\partial_{y_k} - i\kappa H\widetilde{F}_k\big)\psi_l} - \kappa^2|\widetilde{\psi}_l|^2 + \frac{\kappa^2}{2}|\widetilde{\psi}_l|^4\Big]\,dy$$

Inserting the estimates (5.6) and (5.7) we obtain (again it is assumed that δ is sufficiently small)

$$\begin{aligned}\mathcal{E}_0^{\rm 3D}(u_l, \mathbf{F}) \geq{}& (1 - C\delta)\int_{O_\delta}\Big(|(\nabla_y - i\kappa H\widetilde{\mathbf{F}}(y))\psi_l|^2 - \kappa^2|\psi_l|^2 + \frac{\kappa^2}{2}|\psi_l|^4\Big)\,dy \\ &- C\delta\kappa^2\int_{O_\delta}|\psi_l|^2\,dy\,.\end{aligned}$$

Using the pointwise inequality (with $\varepsilon > 0$ arbitrary)

$$|(\nabla_y - i\kappa H\widetilde{\mathbf{F}})\psi_l|^2 \geq (1-\varepsilon)|(\nabla_y - i\kappa H\mathbf{E}_{\nu_l})\psi_l|^2 + (1-\varepsilon^{-1})(\kappa H)^2|(\widetilde{\mathbf{F}} - \mathbf{E}_{\nu_l})\psi_l|^2$$

with $\mathbf{E}_\nu$ from (3.4), we obtain

$$\begin{aligned}\mathcal{E}_0^{3\mathrm{D}}(u_l,\mathbf{F}) \geq (1-C\varepsilon-C\delta)\mathcal{E}_0^{3\mathrm{D}}(\psi_l,\mathbf{E}_{\nu_l}) - C\varepsilon^{-1}(\kappa H)^2\int_{O_\delta}|(\widetilde{\mathbf{F}}-\mathbf{E}_{\nu_l})\psi_l|^2\,dy \\ - C(\varepsilon\kappa^2+\delta\kappa^2)\int_{O_\delta}|\psi_l|^2\,dy\,.\end{aligned}$$

We estimate the integral $\int_{O_\delta}|(\widetilde{\mathbf{F}}-\mathbf{E}_{\nu_l})\psi_l|^2\,dy$ using (5.9). In this way, we find,

$$\varepsilon^{-1}(\kappa H)^2\int_{O_\delta}|(\widetilde{\mathbf{F}}-\mathbf{E}_{\nu_l})\psi_l|^2\,dy \leq C\varepsilon^{-1}(\kappa H)^2\delta^4\int_{O_\delta}|\psi_l|^2\,dy.$$

We conclude that (with the choice $\varepsilon = \kappa\delta^2$)

$$\mathcal{E}_0^{3\mathrm{D}}(u_l,\mathbf{F}) \geq (1-C\delta^2\kappa-C\delta)\mathcal{E}_0^{3\mathrm{D}}(\psi_l,\mathbf{E}_{\nu_l}) - (\delta+\delta^2\kappa)\kappa^2\int_{O_\delta}|\psi_l|^2\,dy\,.$$

After a scaling, $y = (\kappa H)^{-1/2}z$, we obtain

$$\mathcal{E}_0^{3\mathrm{D}}(\psi_l,\mathbf{E}_{\nu_l}) = \frac{1}{\sqrt{\kappa H}}\int_{O_{\sqrt{\kappa H}\delta}}\Big(|(\nabla_z - i\mathbf{E}_{\nu_l}(z))\widetilde{\psi}_l|^2 - \frac{\kappa}{H}|\widetilde{\psi}_l|^2 + \frac{\kappa}{2H}|\widetilde{\psi}_l|^4\Big)\,dz\,, \tag{5.14}$$

where $\widetilde{\psi}_l(z) = \psi_l\big((\kappa H)^{-1/2}z\big)$.

We impose the condition $\sqrt{\kappa H}\delta \gg 1$. Next, by defining $\widetilde{\lambda} = [\kappa/H - 1]_+$, we get $0 < \kappa/H \leq 1+\widetilde{\lambda}$.

Let $\mathfrak{b} = \min\big(\kappa/H, 1\big)$. It is easy to see that,

$$-\frac{1}{\sqrt{\kappa H}}\int_{O_{\sqrt{\kappa H}\delta}}\frac{\kappa}{H}|\widetilde{\psi}_l|^2\,dz \geq -\frac{1}{\sqrt{\kappa H}}\int_{O_{\sqrt{\kappa H}\delta}}\mathfrak{b}|\widetilde{\psi}_l|^2\,dz - C\widetilde{\lambda}(\kappa H)\int_{O_\delta}|\psi_l|^2\,dz\,.$$

We insert this estimate into (5.14), then we apply[2] Theorem 3.9 (with $\ell = \sqrt{\kappa H}\delta$). In this way, we conclude that,

$$\mathcal{E}_0^{3\mathrm{D}}(u_l,\mathbf{F}) \geq (1-C\delta^2\kappa-C\delta)\sqrt{\kappa H}\,E(\mathfrak{b},\nu(x_{0,l}))\,(4\delta^2) - C(\delta+\delta^2\kappa+\widetilde{\lambda})\kappa^2\int_{O_\delta}|\psi_l|^2\,dy\,. \tag{5.15}$$

provided that κ is large enough. We combine the estimates in (5.11)-(5.15) to get,

$$\begin{aligned}\mathcal{E}_0^{3\mathrm{D}}(\chi_1 h\psi,\mathbf{A}) &\geq \sum_l\Big\{\mathcal{E}_0^{3\mathrm{D}}(\widetilde{\chi}_l\chi_1 h\psi,\mathbf{A})\Big\} - C\alpha^{-2}\delta^{-1} \\ &\geq \sum_l\Big\{(1-C\delta^2\kappa-C\delta)\sqrt{\kappa H}E(\mathfrak{b},\nu(x_{0,l}))(4\delta^2) \\ &\qquad\qquad - C(\delta+\delta^2\kappa+\widetilde{\lambda})\kappa^2\int_{O_\delta}|\psi_l|^2\,dy\Big\} \\ &\quad - C\alpha^{-2}\delta^{-1}.\end{aligned}$$

We estimate as before using the finite overlap of the supports of the partition of unity,

$$\sum_\ell\int_{O_\delta}|\psi_l|^2\,dy \leq C\int_{\Omega_1}|\psi|^2\,dx \leq C\delta. \tag{5.16}$$

[2] We need to apply Theorem 3.9 when $\mathfrak{b} \in [\Theta_0, 1]$. If $\mathfrak{b} \in (0, \Theta_0)$, we use Remark 3.5.

Next, we note that we have a Riemann sum,

$$\left|\sum_l \Big\{ E(\mathfrak{b},\nu(x_{0,l}))(4\delta^2)\Big\} - \int_{\overline{D}\cap\partial\Omega} E(\mathfrak{b},\nu(x))\,d\sigma(x)\right| \le C\delta\,.$$

Hence, we find that

$$\begin{aligned}\mathcal{E}_0^{3D}(\chi_1 h\psi,\mathbf{A}) \ge & (1-C\delta^2\kappa - C\delta)\sqrt{\kappa H}\int_{\overline{D}\cap\partial\Omega} E(\mathfrak{b},\nu(x))\,d\sigma(x)\\ & - C\delta\sqrt{\kappa H} - C(\delta^2\kappa+\delta+\widetilde{\lambda})\delta\kappa^2 - C\alpha^{-2}\delta^{-1}.\end{aligned} \tag{5.17}$$

We choose the parameters

$$\delta = \kappa^{-3/4},\qquad \alpha = \kappa^{-1/16}, \tag{5.18}$$

so that the remainder terms in (5.17) are all estimated as

$$o\Big(\max(\kappa,[\kappa-H]_+^2)\Big)\,,$$

as $\kappa\to\infty$.

Proof of (5.13). Let $\widetilde{\mathbf{A}} = \mathbf{A} - \nabla\phi_l$. Using the pointwise inequality

$$|z_1 - z_2|^2 \ge (1-\varepsilon)|z_1|^2 + (1-\varepsilon^{-1})|z_2|^2\,,$$

which is valid for all complex numbers z_1, z_2 and real numbers $\varepsilon\in(0,1)$, we obtain the lower bound

$$\mathcal{E}^{3D}(u_l,\widetilde{\mathbf{A}}) \ge (1-\varepsilon)\mathcal{E}_0^{3D}(u_l,\mathbf{F}) + \varepsilon^{-1}(\kappa H)^2\int_\Omega |(\widetilde{\mathbf{A}}-\mathbf{F})u_l|^2\,dx - \varepsilon\int_\Omega \kappa^2|u_l|^2\,dx. \tag{5.19}$$

Observing that

$$(\widetilde{\mathbf{A}}-\mathbf{F})(x) = (\mathbf{A}-\mathbf{F})(x) - (\mathbf{A}-\mathbf{F})(x_{0,l})\,,$$

we get from Corollary 4.4

$$\begin{aligned}(\kappa H)^2\int_\Omega |(\widetilde{\mathbf{A}}-\mathbf{F})u_l|^2\,dx &\le C\lambda^{1/3}\kappa^2\int |x-x_{0,l}|^2\,|u_l|^2\,dx\\ &\le C\lambda^{1/3}\kappa^2\delta^2\int |u_l|^2\,dx\,.\end{aligned}$$

Inserting this into (5.19), and making for simplicity the non-optimal choice $\varepsilon=\delta$, we finish the proof of (5.13) by observing that $\lambda\le 1$ for large values of κ. □

5.4. **The bulk term.** We return to the energy decomposition in (5.4). The choice we made for the parameter δ allows us to estimate the upper bound in (5.5) as $\mathcal{O}(\kappa^{3/4}) = o(\kappa)$. The surface term in (5.4) is estimated using (5.17). It only remains to estimates the bulk term $\mathcal{E}_0(\chi_2 h\psi,\mathbf{A})$ appearing in (5.4). The estimate of this term was the objective of the first part of this paper [12]. Actually, Theorem 6.1 of [12] tells us that,

$$\mathcal{E}_0(\chi_2 h\psi,\mathbf{A}) \ge E_2|D|\,[\kappa-H]_+^2 + o\Big(\max(\kappa,[\kappa-H]_+^2)\Big)\,.$$

Inserting this estimate into (5.4), then using the lower bound in (5.17) and the choice of δ finishes the proof of Theorem 5.1.

6. Upper bound

The aim of this section is to give an asymptotic upper bound of the ground state energy in (1.4). This will be done through the computation of the energy of relevant test configurations, whose construction hints at the expected behavior of the actual minimizers of the energy in (1.1).

The main theorem of this section is stated below.

Theorem 6.1. *For all $\tilde{a}(\kappa) = o(1)$ as $\kappa \to \infty$ and $C > 0$, there exists $0 < \mathrm{err}(\kappa) = o(1)$ such that if*

$$1 - \tilde{a}(\kappa) \leq \frac{H}{\kappa} \leq C.$$

Then, as $\kappa \to \infty$, the ground state energy in (1.4) satisfies,

$$\mathrm{E}_{g.st}(\kappa, H) \leq \sqrt{\kappa H} \int_{\partial\Omega} E(\mathfrak{b}, \nu(x))\, d\sigma(x) + E_2|\Omega|\, [\kappa - H]_+^2 + \mathrm{err}\left(\max(\kappa, [\kappa - H]_+^2)\right).$$

Here $\mathfrak{b} = \min\left(\kappa/H, 1\right)$, and $d\sigma(x)$ is the surface measure on the boundary of Ω.

Proof of Theorem 6.1.

Boundary trial configuration

Let $\delta > 0$ be small but fixed. We will choose another parameter $\eta > 0$ which will be specified as a negative power of κ below.

Choose a finite collection of points $\{x_j\} \subset \partial\Omega$ such that

$$\forall j \neq k: \quad \delta/2 \leq \mathrm{dist}(x_j, x_k) \qquad \text{and} \qquad \forall j: \quad \min_{k \neq j} \mathrm{dist}(x_j, x_k) \leq 2\delta.$$

Define U_j as

$$U_j = \{x \in \partial\Omega \ : \ \forall\, k \neq j\,,\ \mathrm{dist}(x, x_j) < \mathrm{dist}(x, x_k)\}\,.$$

Clearly the U_j's are disjoint and $\partial\Omega = \bigcup_j \overline{U}_j$.

Next, we construct a family of sets that covers a tubular neighborhood of $\partial\Omega$. That will be done by using the boundary coordinates (y_1, y_2, y_3) introduced in Sec. 5.2.1 ($y_1 = 0$ defines the corresponding part in $\partial\Omega$). Let Φ_j be the coordinate transformation that straightens a neighborhood V_j of the point x_j such that $\Phi_j(x_j) = 0$. We may assume that $U_j \subset V_j$ for all j. Let

$$O_j = \{x = \Phi_j^{-1}(y_1, y_2, y_3) \ : \ \Phi_j^{-1}(0, y_2, y_3) \in U_j \text{ and } 0 < y_1 < \eta\}\,.$$

We now choose a parameter δ':

$$\frac{1}{\sqrt{\kappa H}} \ll \delta' \ll \delta$$

(δ' will be chosen below as a negative power of κ). Define $\widetilde{O}_j^{2\mathrm{D}} = \Phi_j(O_j) \cap \{y \in \mathbb{R}^3 \ : \ y_1 = 0\}$. We may cover $\widetilde{O}_j^{2\mathrm{D}}$ by a square lattice $\{K_{j,i}\}$ where each $K_{j,i}$ is centered at point $y_{j,i}$ and has side-length $2\delta'$. Let

$$\mathcal{J}_j = \{i \ : \ K_{j,i} \subset \widetilde{O}_j^{2\mathrm{D}}\}, \quad N_j = \mathrm{Card}\,\mathcal{J}_j\,,$$

and $N = \sum_j N_j$. Clearly, the number N satisfies,

$$N \times (2\delta')^2 \to |\partial\Omega| \quad \text{as } \delta' \to 0\,. \tag{6.1}$$

We combine the coordinate transformation Φ_j by a translation so that the new coordinates of the point $y_{j,i}$ are 0. Thus, we let $\Phi_{j,i}$ be the resulting coordinate transformation valid in $\Phi_j^{-1}(\mathbb{R}^+ \times K_{j,i}) \cap V_j$ such that $\Phi_{j,i}(y_{j,i}) = 0$.

We consider only indices $i \in \mathcal{J}_j$ for some j. Let $x_{j,i} = \Phi_{j,i}^{-1}(0)$. At each point $x_{j,i}$, the magnetic field $\beta = (0,0,1)$ forms an angle $\nu_{j,i} = \nu(x_{j,i}) \in [0, \pi/2]$ with the tangent plane to $\partial\Omega$. As explained earlier in Sec. 5.2.1, there exists a real valued function $\phi_{j,i}$ such that, if $\mathbf{F}$ is

the vector field defined in cartesian coordinates by $\mathbf{F}(x_1,x_2,x_3)=(-x_2/2,x_1/2,0)$, and $\widetilde{\mathbf{F}}$ is the vector field defined in y-coordinates by the relation in (5.8), then,

$$\widetilde{\mathbf{F}}+\nabla\phi_{j,i}=\mathbf{E}_{\nu_{j,i}}+\mathbf{R}_{j,i}\,,$$

where $\mathbf{E}_{\nu_{j,i}}$ is the magnetic potential from (3.4), and $\mathbf{R}_{j,i}$ is a vector field given in y-coordinates by,

$$\mathbf{R}_{j,i}=\begin{pmatrix}0\\ \mathcal{O}(|y_1|^2+|y_2|^2+|y_3|^2)\\ \mathcal{O}(|y_1|^2+|y_2|^2+|y_3|^2)\end{pmatrix}.$$

Let $\mathfrak{b}=\min(\kappa/H,1)$, $\ell=\delta'\sqrt{\kappa H}$ and $u_{j,i}$ a minimizer of the functional $\mathcal{G}_{\mathfrak{b},\nu_{j,i},\ell}$ introduced in (3.11). For $x=\Phi^{-1}_{j,i}(y_1,y_2,y_3)\in O_j$, we put,

$$\psi_{j,i}(x)=\begin{cases}e^{-i\kappa H\phi_{j,i}}u_{j,i}(y\sqrt{\kappa H}\,), & x=\Phi^{-1}_{j,i}(y_1,y_2,y_3)\in O_j\\ 0, & \text{else}\end{cases}.$$

Notice that by construction the supports of the $\psi_{j,i}$ do not overlap.

We define a test function $\psi_{\rm bnd}\in H^1(\Omega;\mathbb{C})$ as follows,

$$\forall\, x\in\Omega\,,\quad \psi^{\rm bnd}_{\eta,\delta}(x)=h\left(\frac{\operatorname{dist}(x,\partial\Omega)}{\eta}\right)\psi(x)\,,\tag{6.2}$$

where h is a cut-off function satisfying,

$$\operatorname{supp}h\subset[-1,1]\,,\quad 0\le h\le 1 \text{ in }\mathbb{R}\,,\quad h(x)=1\text{ in }[-1/2,1/2]\,,$$

and

$$\psi=\sum_{i,j}\psi_{j,i}.$$

Energy of the test-configuration

We will compute the energy of the configuration $(\psi^{\rm bnd}_{\eta,\delta},\mathbf{F})$. Notice that the construction of $\psi^{\rm bnd}_{\eta,\delta}$ and the change of variable formulas in Sec. 5.2.1 together imply the existence of a constant $C>0$ (independent of δ) such that, if κ is sufficiently large and β is an arbitrary real number in $(0,1)$, then we have the upper bound,

$$\begin{aligned}\mathcal{E}^{\rm 3D}(\psi^{\rm bnd}_{\eta,\delta},\mathbf{F})\le{}&(1+C(\delta+\eta+\beta))\sum_{j,i}\int_{V_{\delta'}}|(\nabla-i\kappa H\mathbf{E}_{\nu_{j,i}})\,h(y_1/\eta)\,u_{j,i}(y\sqrt{\kappa H})|^2\,dy\\ &+C\kappa^2H^2\beta^{-1}(\delta'^4+\eta^4)\sum_{j,i}\int_\Omega|\psi_{j,i}(x)|^2\,dx+\sum_{j,i}\int_\Omega\left(-\kappa^2|\psi_{j,i}|^2+\frac{\kappa^2}{2}|\psi_{j,i}|^4\right)dx\,,\end{aligned}\tag{6.3}$$

where $V_{\delta'}=(0,\infty)\times(-\delta',\delta')\times(-\delta',\delta')$.

By Theorem 3.6

$$\int\left(|u_{j,i}(z)|^2+|u_{j,i}(z)|^4\right)dz\le C\ell^2\,,$$

where C is a universal constant and $\ell=\delta'\sqrt{\kappa H}$. We use the change of variable formulae in Sec. 5.2.1 to express

$$-\int|\psi_j|^2\,dx+\frac12\int|\psi_j|^4\,dx$$

in boundary coordinates, then we apply the change of variable $z=y\sqrt{\kappa H}$ and use the decay of $u_{j,i}$. In this way, we get a constant C such that, for all j and $i\in\mathcal{J}_j$, we have,

$$-\kappa^2\int|\psi_{j,i}|^2\,dx+\frac{\kappa^2}{2}\int|\psi_{j,i}|^4\,dx\le\sqrt{\kappa H}\int_{U_\ell}\left(-\frac{\kappa}{H}|u_{j,i}(z)|^2+\frac{\kappa}{2H}|u_{j,i}|^4\right)dx+\frac{C\kappa^2}{\sqrt{\kappa H}}(\delta+\eta)(\delta')^2\,.$$

Here $U_\ell = (0,\infty)\times(-\ell,\ell)\times(-\ell,\ell)$. Also,

$$C\kappa^2H^2\beta^{-1}(\delta'^4+\eta^4)\int_\Omega|\psi_{j,i}(x)|^2\,dx \le C\kappa^2\frac{H^2}{\sqrt{\kappa H}}\beta^{-1}(\delta'^4+\eta^4)(\delta')^2.$$

Using again Theorem 3.6

$$\begin{aligned}
&\int_{V_{\delta'}}|(\nabla - i\kappa H\mathbf{E}_{\nu_{j,i}})\,h(y_1/\eta)\,u_{j,i}(y\sqrt{\kappa H})|^2\,dy\\
&\le (1+\beta)\int_{V_{\delta'}}|(\nabla - i\kappa H\mathbf{E}_{\nu_{j,i}})u_{j,i}(y\sqrt{\kappa H})|^2\,dy + \beta^{-1}\eta^{-2}\int_{V_{\delta'}}[h'(y_1/\eta)]^2\,|u_{j,i}(y\sqrt{\kappa H})|^2\,dy\\
&\le (1+\beta)\sqrt{\kappa H}\int_{U_\ell}|(\nabla - i\mathbf{E}_{\nu_{j,i}})u_{j,i}|^2\,dz + \beta^{-1}\eta^{-3}(\kappa H)^{-2}\int_{U_\ell}z_1|u_{j,i}|^2\,dz\\
&\le (1+\beta)\sqrt{\kappa H}\int_{U_\ell}|(\nabla - i\mathbf{E}_{\nu_{j,i}})u_{j,i}|^2\,dz + \beta^{-1}\eta^{-3}(\kappa H)^{-1}(\delta')^2.
\end{aligned}$$

Substituting the above upper bounds into (6.3), we get,

$$\begin{aligned}
&\mathcal{E}^{3\mathrm{D}}(\psi^{\mathrm{bnd}}_{\eta,\delta},\mathbf{F})\\
&\le \sum_{j,i}\Bigg\{[1+C(\delta+\eta+\beta)]\sqrt{\kappa H}\int_{U_\ell}\Big(|(\nabla - i\mathbf{E}_{\nu_{j,i}})u_{j,i}|^2 - \frac{\kappa}{H}|u_{j,i}|^2 + \frac{\kappa}{2H}|u_{j,i}|^4\Big)\,dz\\
&+ C\kappa^2\frac{H^2}{\sqrt{\kappa H}}\beta^{-1}(\delta'^4+\eta^4)(\delta')^2 + C\kappa^2\frac{1}{\sqrt{\kappa H}}(\delta+\eta+\beta)(\delta')^2 + C\beta^{-1}\eta^{-3}(\kappa H)^{-1}(\delta')^2\Bigg\}.
\end{aligned}\tag{6.4}$$

Recall that $\mathfrak{b}=\min(\kappa/H,1)$. By our choice of $u_{j,i}$, we get that,

$$\int_{U_\ell}\Big(|(\nabla - i\mathbf{E}_{\nu_{j,i}})u_{j,i}|^2 - \frac{\kappa}{H}|u_{j,i}|^2 + \frac{\kappa}{2H}|u_{j,i}|^4\Big)\,dz \le (2\delta')^2\left[E(\mathfrak{b},\nu_{j,i})+o(1)\right].$$

Also, by (6.1), we have

$$\sum_{i,j}(\delta')^2\le C.$$

Inserting this and the bounds on H/κ into (6.4), we get

$$\begin{aligned}
\mathcal{E}^{3\mathrm{D}}(\psi^{\mathrm{bnd}}_{\eta,\delta},\mathbf{F}) &\le [1+C(\delta+\eta+\beta)]\sqrt{\kappa H}\sum_{j,i}(2\delta')^2\left[E(\mathfrak{b},\nu_{j,i})+o(1)\right]\\
&\quad + C\big[\kappa^3\beta^{-1}(\delta'^4+\eta^4)+\kappa(\delta+\eta+\beta)+\beta^{-1}\eta^{-3}\kappa^{-2}\big].
\end{aligned}\tag{6.5}$$

We can now, for example, choose

$$\eta=\delta'=\kappa^{-7/12},\qquad \beta=\kappa^{-1/6}.$$

The sum is a Riemann sum, hence as $\delta'\to 0$,

$$\sum_{j,i}(2\delta')^2E(\mathfrak{b},\nu_{j,i}) = \int_{\partial\Omega}E(\mathfrak{b},\nu(x))\,d\sigma(x)+o(1).$$

Therefore, we get the following upper bound,

$$\mathcal{E}^{3\mathrm{D}}(\psi^{\mathrm{bnd}}_{\eta,\delta},\mathbf{F}) \le (1+C\delta)\sqrt{\kappa H}\int_{\partial\Omega}E(\mathfrak{b},\nu(x))\,d\sigma(x)+o(\kappa)+C\delta\kappa.\tag{6.6}$$

Since $\delta>0$ was arbitrary this is consistent with the boundary part of the upper bound in Theorem 6.1.

Bulk trial configuration
We keep the choice of the parameters δ and η introduced in the preceding section. Let $\psi^{\rm blk}_{\eta,R}$ be the test function defined in [12, Eq. (6.15)]. The function $\psi^{\rm blk}_{\eta,R}$ vanishes in

$$\{x\in\Omega~:~{\rm dist}(x,\partial\Omega)\leq\eta\}\,.$$

Thus, $\psi^{\rm blk}_{\eta,R}$ and $\psi^{\rm bnd}_{\eta,\delta}$ have disjoint support. Consequently, by defining,

$$f(x)=\psi^{\rm blk}_{\eta,R}(x)+\psi^{\rm bnd}_{\eta,\delta}(x)\,,\quad x\in\Omega\,,$$

we get,

$$\mathcal{E}^{\rm 3D}(f,\mathbf{F})=\mathcal{E}^{\rm 3D}(\psi^{\rm blk}_{\eta,R},\mathbf{F})+\mathcal{E}^{\rm 3D}(\psi^{\rm bnd}_{\eta,\delta},\mathbf{F})\,. \tag{6.7}$$

Theorem 6.5 in [12] tells us that

$$\mathcal{E}^{\rm 3D}(\psi^{\rm blk}_{\eta,R},\mathbf{F})\leq E_2|\Omega|\,[\kappa-H]_+^2+o\Big(\max\big(\kappa,[\kappa-H]_+^2\big)\Big)\,.$$

Inserting this estimate and that in (6.6) into (6.7), we get,

$$\mathcal{E}^{\rm 3D}(f,\mathbf{F})\leq\sqrt{\kappa H}\int_{\partial\Omega}E(\mathfrak{b},\nu(x))\,d\sigma(x)+E_2|\Omega|\,[\kappa-H]_+^2+o\Big(\max\big(\kappa,[\kappa-H]_+^2\big)\Big)\,.$$

Recalling the ground state energy in (1.4), we deduce the following upper bound,

$$\mathrm{E}_{\rm g.st}(\kappa,H)\leq\sqrt{\kappa H}\int_{\partial\Omega}E(\mathfrak{b},\nu(x))\,d\sigma(x)+E_2|\Omega|\,[\kappa-H]_+^2+o\Big(\max\big(\kappa,[\kappa-H]_+^2\big)\Big)\,.$$

This finishes the proof of Theorem 6.1. ☐

7. Proof of main theorems

7.1. **Proof of Theorem 1.1.** We combine the lower bound of Theorem 5.1 (with $D=\Omega$ and $h=1$) and the upper bound of Theorem 6.1.

7.2. **Proof of Theorem 1.2.** Let $D\subset\Omega$ be open and smooth. Let $\mu:\mathbb{R}_+\to\mathbb{R}$ be a function satisfying $\lim_{\kappa\to\infty}\mu(\kappa)=0$. Suppose that the magnetic field H satisfies $H\geq\kappa-\mu(\kappa)\kappa$.

Using Theorem 3.3 in [12], we know that,

$$\|\psi\|_{L^\infty(\omega_\kappa)}=o(1)\,,\quad\text{as }\kappa\to\infty\,, \tag{7.1}$$

where

$$\omega_\kappa=\{x\in\Omega~:~{\rm dist}(x,\partial\Omega)\geq g_1(\kappa)/\kappa\}\,,$$

and $g_1(\kappa)$ is any function satisfying $\lim_{\kappa\to\infty}g_1(\kappa)=\infty$.

Lemma 7.1. *If $(\psi,\mathbf{A})$ is a solution of (1.3), then,*

$$\mathcal{E}(\psi,\mathbf{A};D)-\mathcal{E}(f\psi,\mathbf{A};D)-\int_D|\nabla f|^2|\psi|^2\,dx+\frac{\kappa^2}{2}\int_D(1-f^2)^2|\psi|^4\,dx$$
$$=-\,{\rm Re}\int_{\partial\Omega}|\psi|^2\overline{f}\;\nu\cdot\nabla f\,d\sigma+o(\kappa)\,. \tag{7.2}$$

holds true as $\kappa\to\infty$. Here ν is the unit inward normal vector of $\partial\Omega$ and $f\in H^1(\Omega)$ is any function such that,

$$\nabla f\in L^\infty(\Omega)\,,\quad{\rm supp}\,f\subset\overline{D}\,.$$

Proof. Using the estimate in (7.1), the result of the lemma follows through an integration by parts applied to the term $\mathcal{E}(f\psi,\mathbf{A};D)$. See [13, Lemma 6.1] for details. ☐

Lemma 7.2. *If $(\psi,\mathbf{A})$ is a minimizer of the functional in (1.1), then as $\kappa\to\infty$,*

$$\kappa^2H^2\int_\Omega|\,{\rm curl}(\mathbf{A}-\mathbf{F})|^2\,dx=o\big(\max(\kappa,[\kappa-H]_+^2)\big)\,.$$

Proof. Recall the functional $\mathcal{E}_0$ in (5.2). Theorem 5.1 tells us that,

$$\mathcal{E}_0(\psi,\mathbf{A}) \geq \int_{\partial\Omega} E(\mathfrak{b},\nu(x))\,d\sigma(x) + E_2|\Omega|\,[\kappa-H]_+^2 + o\big(\max(\kappa,[\kappa-H]_+^2)\big)\,.$$

Consequently, the result of the lemma follows by observing that

$$\mathcal{E}(\psi,\mathbf{A}) = \mathcal{E}_0(\psi,\mathbf{A}) + \kappa^2H^2\int_{\mathbb{R}^3}|\operatorname{curl}(\mathbf{A}-\mathbf{F})|^2\,dx$$

and using the upper bound obtained in Theorem 6.1. □

Proof of (1.8). Let $(\psi,\mathbf{A})$ be a solution of (1.3). Multiplying the first equation in (1.3) by $\overline{\psi}$ then integrating over D we get,

$$\int_D |(\nabla - i\kappa H\mathbf{A})\psi|^2\,dx + \int_{\partial D}\overline{\psi}\,\nu\cdot(\nabla - i\kappa H)\psi\,d\sigma(x) = \kappa^2\int_D(1-|\psi|^2)|\psi|^2\,dx\,. \tag{7.3}$$

Using (7.1), we can show that (see [13, Lemma 6.1] for details),

$$\int_{\partial D}\overline{\psi}\,\nu\cdot(\nabla - i\kappa H)\psi\,d\sigma(x) = o(\kappa)$$

as $\kappa\to\infty$. Consequently, we may rewrite (7.3) as follows,

$$\mathcal{E}_0(\psi,\mathbf{A};D) = -\frac{\kappa^2}{2}\int_D|\psi|^4\,dx + o(\kappa)\,, \tag{7.4}$$

where $\mathcal{E}_0$ is the functional introduced in (5.2).

Let $h\in C_c^\infty(\mathbb{R})$ be a cut-off function such that

$$\operatorname{supp} h\subset[-1,1]\,,\quad h(x)=1 \text{ in } [-1/2,1/2]\,,\quad 0\leq h\leq 1 \text{ in } \mathbb{R}\,.$$

We define a function f as follows,

$$\forall\, x\in\Omega\,,\quad f(x) = 1 - h\left(\frac{\operatorname{dist}(x,D^{\mathrm{c}})}{L^{-1}}\right)\,, \tag{7.5}$$

where $D^{\mathrm{c}} = \Omega\setminus D$ is the complement in Ω of D, and

$$L = \max(\kappa,[\kappa-H]_+^2)\,. \tag{7.6}$$

We will prove the estimate below,

$$-\int_D|\nabla f|^2|\psi|^2\,dx + \frac{\kappa^2}{2}\int_D(1-f^2)^2|\psi|^4\,dx + \operatorname{Re}\int_{\partial\Omega}|\psi|^2\overline{f}\,\nu\cdot\nabla f\,d\sigma = o\big(\max(\kappa,[\kappa-H]_+^2)\big)\,. \tag{7.7}$$

Details concerning the derivation of the estimate in (7.7) will be postponed to the end of this proof.

Using (7.7) and the decomposition formula of Lemma 7.1, we get,

$$\mathcal{E}_0(\psi,\mathbf{A};D) = \mathcal{E}_0(f\psi,\mathbf{A};D) + o\big(\max(\kappa,[\kappa-H]_+^2)\big)\,. \tag{7.8}$$

We apply Theorem 5.1 to bound $\mathcal{E}_0(f\psi,\mathbf{A};D)$ from below. In this way, we infer from (7.8),

$$\mathcal{E}_0(\psi,\mathbf{A};D) \geq \sqrt{\kappa H}\int_{\overline{D}\cap\partial\Omega} E(\mathfrak{b},\nu(x))\,d\sigma(x) + E_2|D|\,[\kappa-H]_+^2 + o\big(\max(\kappa,[\kappa-H]_+^2)\big)\,. \tag{7.9}$$

Inserting this lower bound into (7.4) finishes the proof of (1.8). The only point left is the justification of the estimate in (7.7).

Proof of (7.7)*:* Let $g_1(\kappa) = L^{1/2}$. Recall the estimate in (7.1) valid in the set ω_κ. In order to estimate the integral $\displaystyle\int_D (1-f^2)^2|\psi|^4\,dx$, we use the simple decomposition,

$$\int_D (1-f^2)^2|\psi|^4\,dx = \int_{\omega_\kappa\cap D}(1-f^2)^2|\psi|^4\,dx + \int_{\omega_\kappa^c\cap D}(1-f^2)^2|\psi|^4\,dx\,.$$

We estimate the integral over $\omega_\kappa\cap D$ as follows:

$$\int_{\omega_\kappa\cap D}(1-f^2)^2|\psi|^4\,dx \le o\Big(\int_D (1-f^2)^2\,dx\Big) \le o(L^{-1}). \tag{7.10}$$

It is left to estimate the integral over $\omega_\kappa^c\cap D$. Here we use that the measure of the set

$$\mathrm{supp}(1-f^2)\cap\omega_\kappa^c\cap D \subseteq \{x\in\Omega\ :\ \mathrm{dist}(x,\partial\Omega)\le g_1(\kappa)/\kappa\,,\ \mathrm{dist}(x,D^c)\le 2L^{-1}\}$$

is

$$\mathcal{O}\left(\frac{g_1(\kappa)}{\kappa}\times 2L^{-1}\right)\,.$$

As a consequence, we get by using the bound $\|\psi\|_{L^\infty(\Omega)}\le 1$ together with our choice of $g_1(\kappa) = L^{1/2}$ that,

$$\int_{\omega_\kappa^c\cap D}(1-f^2)^2|\psi|^4\,dx = \mathcal{O}\big(L^{-1/2}\kappa^{-1}\big)\,.$$

The estimates obtained for the integrals over $\omega_\kappa\cap D$ and $\omega_\kappa^c\cap D$ together yield that,

$$\kappa^2\int_D (1-f^2)^2|\psi|^4\,dx = o(\kappa)\,. \tag{7.11}$$

In a similar fashion, we may show that,

$$\int_D |\nabla f|^2|\psi|^2\,dx = o(L)\,,\quad \mathrm{Re}\int_{\partial\Omega}|\psi|^2\overline{f}\,\nu\cdot\nabla f\,d\sigma = o(L)\,.$$

The two aforementioned estimates above, together with that in (7.11) and the definition of $L=\max(\kappa,[\kappa-H]_+^2)$, yield the estimate in (7.7). □

Proof of (1.9). Suppose that $(\psi,\mathbf{A})$ is a minimizer of the functional in (1.1). Let the function f be as in (7.5). Since $(\psi,\mathbf{A})$ is a solution of (1.3), then the formulas in (7.4) and (7.8) still hold true.

We define a function $g\in H^1(\Omega)$ as follows,

$$\forall\ x\in\Omega\,,\quad g(x) = \sqrt{1-f^2(x)}\,,$$

so that $f^2(x)+g^2(x)=1$ in Ω. Consequently, we have the decomposition formula,

$$\mathcal{E}(\psi,\mathbf{A}) \ge \mathcal{E}(f\psi,\mathbf{A}) + \mathcal{E}(g\psi,\mathbf{A}) - \int_\Omega \left(|\nabla f|^2+|\nabla g|^2\right)|\psi|^2\,dx\,. \tag{7.12}$$

Is results from Lemma 4.1 that as $\kappa\to\infty$,

$$\int_\Omega (|\nabla f|^2+|\nabla g|^2)|\psi|^2\,dx = o\big(\max(\kappa,[\kappa-H]_+^2)\big)\,. \tag{7.13}$$

Also, using Theorem 5.1, we obtain the lower bound,

$$\mathcal{E}(g\psi,\mathbf{A}) \ge \sqrt{\kappa H}\int_{\overline{D^c}\cap\partial\Omega} E(\mathfrak{b},\nu(x))\,d\sigma(x) + E_2|D^c|\,[\kappa-H]_+^2 + o\big(\max(\kappa,[\kappa-H]_+^2)\big)\,.$$

We insert this estimate together with that in (7.13) into (7.12). Also, we use the upper bound for $\mathcal{E}(\psi,\mathbf{A})$ obtained in Theorem 6.1. In this way we infer from (7.12),

$$\mathcal{E}(f\psi,\mathbf{A}) \le \sqrt{\kappa H}\int_{\overline{D}\cap\partial\Omega} E(\mathfrak{b},\nu(x))\,d\sigma(x) + E_2|D|\,[\kappa-H]_+^2 + o\big(\max(\kappa,[\kappa-H]_+^2)\big)\,. \tag{7.14}$$

Using the estimate of Lemma 7.2, we get further,

$$\mathcal{E}_0(f\psi,\mathbf{A})\leq\sqrt{\kappa H}\int_{\overline{D}\cap\partial\Omega}E(\mathfrak{b},\nu(x))\,d\sigma(x)+E_2|D|\,[\kappa-H]_+^2+o\big(\max(\kappa,[\kappa-H]_+^2)\big)\,.$$

We insert this estimate into (7.8). This gives us an upper bound of $\mathcal{E}_0(\psi,\mathbf{A};D)$, which when inserted into (7.4) yields the upper bound,

$$-\frac{\kappa^2}{2}\int_D|\psi|^4\,dx\leq\sqrt{\kappa H}\int_{\overline{D}\cap\partial\Omega}E(\mathfrak{b},\nu(x))\,d\sigma(x)+E_2|D|\,[\kappa-H]_+^2+o\big(\max(\kappa,[\kappa-H]_+^2)\big)\,.$$

Combining this estimate with (1.8) finishes the proof of (1.9). □

Acknowledgements

The authors were partially supported by the Lundbeck foundation and the European Research Council under the European Community's Seventh Framework Programme (FP7/2007-2013)/ERC grant agreement n° 202859. AK was partially supported by a grant from the Lebanese University.

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